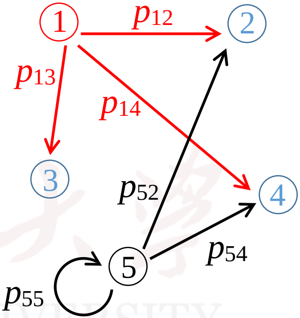
第一章、马氏链

§1.1 定义与例子

一、定义

- $\{X_n : n \in \mathbb{Z}_+\} = \{X_n\}$ 的运动机制:
  每个位置上放一个色子, $\cdots\cdots$
- 转移概率p_{ij}, $p_{i,j}$:
  - 位置i 上的色子: $\{p_{ij} : j \in S\}$.
  - (1) $p_{ij} \geqslant 0, \forall i, j$;
    (2) $\sum_j p_{ij} = 1, \forall i$.



- 固定i. 将分布列 $\{p_{ij} : j \in S\}$ 视为行向量.
- 转移(概率)矩阵:
  $\mathbf{P} = (p_{ij})_{i,j \in S} = (p_{ij})_{S \times S} = (p_{ij})$.

定义 (马氏链(定义1.1.1))

若对任意$n \geqslant 0; i, j, i_0, \cdots, i_{n-1} \in S$ 都有

$$\underset{\sim\sim\sim}{P(X_{n+1} = j | X_n = i, X_0 = i_0, \cdots, X_{n-1} = i_{n-1})} = p_{ij},$$

则称$\{X_n\}$ 是(时齐的)马氏链(*Markov chain*).

- 注：谈论$P(B|AC)$ 时，仅考虑$P(AC) > 0$ 的情形.
- 注：**时齐**指转移概率不随n 改变.
- 注：**非时齐**的马氏链，$\underset{\sim}{\star\star} = p_{ij}(n)$.
- 注：可以是有限长度$\{X_n : 0 \leqslant n \leqslant N\}$.
- 马氏性：$\underset{\sim}{\star\star} = P(*|*)$;

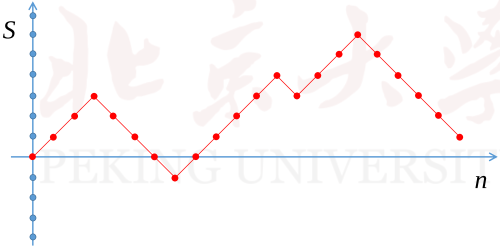
  重点：$\underset{\sim}{\star\star}$ 不依赖于$i_0, \cdots, i_{n-1}$.

二、例子

例1.1.1. 随机游动、步长.

- 一维简单随机游动(SRW). $S = \mathbb{Z}$. $p_{i,i+1} = p_{i,i-1} = \frac{1}{2}$.
- 转移概率图/简图:

- 轨道图:



- 实现: $\xi_1, \xi_2, \cdots$ i.i.d., 等可能取± 1. 令$S_0 = 0$;

$$S_n := S_0 + \xi_1 + \cdots + \xi_n = S_{n-1} + \xi_n.$$

- 注: S_0 也可以为随机变量, 但必须与$(\xi_1, \xi_2, \cdots)$ 独立.

相关模型:

- 结构: $\xi_1, \xi_2, \cdots$ i.i.d., $S_n := S_0 + \xi_1 + \cdots + \xi_n$.
  称如上定义的$\{S_n\}$ 为随机游动, 称$\xi_1, \xi_2, \cdots$ 为步长,
  称$\xi_1 \triangleq \xi$ 的分布为步长分布.

- (一维)紧邻随机游动: $p_{i,i+1} = 1 - p_{i,i-1} = p$.

$$P(\xi = 1) = p, \quad P(\xi = -1) = 1 - p.$$

- d 维随机游动: ξ 取值于$\mathbb{Z}^d$. 例, $d = 1$.

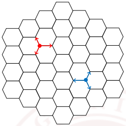
$p_{i,i+k} = P(\xi = k)$

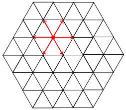
- d 维简单随机游动: 将第i 个坐标取1, 其他坐标取0 的d 维向量记为e_i. 取

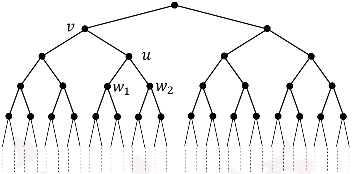
$$P(\xi = e_i) = P(\xi = -e_i) = \frac{1}{2d}, \quad i = 1, \cdots, d.$$

相关模型(续): 例1.1.10. 图上的随机游动.

- (简单)图$G=(V,E)$.





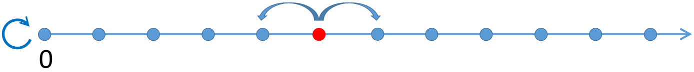


- 顶点/结点、边、邻居(相邻, $u\sim v$)、度($d_u=3$).
- 路径、连通, 例. 连接v, w_1 的路径的长度为2.
- d 维格点($\mathbb{Z}^d$, 距离为1 则连边)、六边形平铺、三角形平铺.
- (带根点的)树、规则树/齐次树$\mathbb{T}^d$、父子关系、叶子.

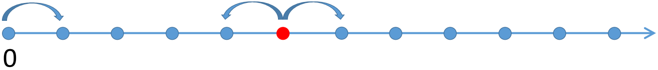
  例. v 在第1 层, $|v|=1$; u 在第2 层, $|u|=2$.
- 随机游动: $S=V$. $p_{ij}=\begin{cases}1/d_i, & 若 j\sim i,\\ 0, & 否则.\end{cases}$

相关模型(续): §1.2 习题3.

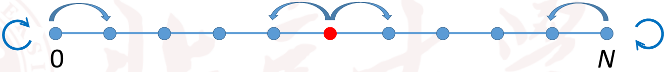
- 吸收壁.



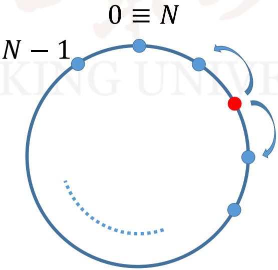
- 反射壁.



- 区间上.

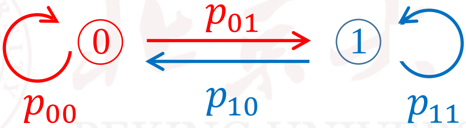


- 离散圆圈/环面上.



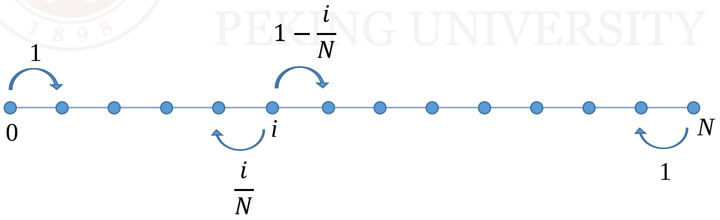
例1.1.7. 两状态马氏链.

- $S = \{0, 1\}$. p_{01}, p_{10}.

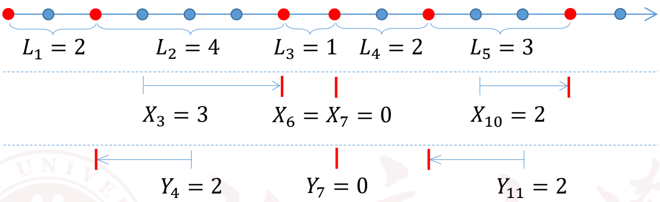


例1.1.8. Ehrenfest模型.

- N 个球, 两个纸箱A, B.
  每次独立地随机选一个球, 把它换到另一个纸箱中.
- X_n: n 次操作后纸箱A 的状态, 即, 其中球的个数.
  则$\{X_n\}$ 是马氏链.
- $S=\{0,1,\cdots,N\}$. $p_{i,i-1}=\frac{i}{N}$, $p_{i,i+1}=1-\frac{i}{N}$.
  转移概率图:



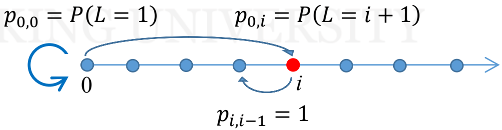
例1.1.9. 更新过程. $L_1, L_2, \cdots$ i.i.d., 取正整数.



- 更新过程$\{X_n\}$: X_n 为时刻n (旧)灯泡的余寿.

  状态空间: $S = \mathbb{Z}_+$.

  转移概率图:

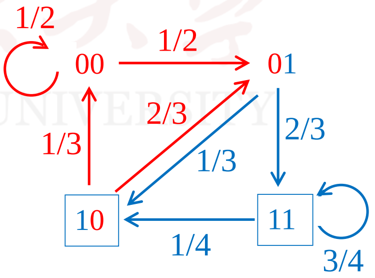


- 老化过程$\{Y_n\}$: Y_n 为时刻n (新)灯泡已经使用的时间.

  状态空间: $S = \mathbb{Z}_+$. 求: 转移概率.(§1.1 习题4)

§1.1 习题5. 投球成功的概率取决于前两次的投球成绩.
如果两次都成功(失败), 则下次投球成功的概率为3/4 (1/2);
如果有一次成功一次失败, 则下次投球成功的概率为2/3.
(1) 试用马氏链来刻画该球员连续投球的过程.

- 第n 次: $\xi_n = 1$ (成功), 0 (失败).
  $\{\xi_n\}$ 不是马氏链!
- 令$X_n = \xi_n\xi_{n+1}$.
  则$\{X_n\}$ 是马氏链.
  $S = \{00, 01, 10, 11\}$.
  转移概率图:



- 令$Y_n = \xi_{2n}\xi_{2n+1}$ 亦可.

三、有限维联合分布

- 回顾定义:

  $P(X_{n+1}=j|X_n=i, X_0=i_0,\cdots,X_{n-1}=i_{n-1}) \overset{(\star)}{=} p_{ij}.$

- 设初分布为μ, 即$X_0\sim\mu$. 则由乘法公式,

$$
\begin{aligned}
&P(\underline{X_0=i_0}, \underset{\sim\sim\sim}{X_1=i_1},\cdots,X_n=i_n)\\
=&P(\underline{A_0})P(\underset{\sim}{A_1}|\underline{A_0})P(A_2|A_0A_1)\cdots P(A_n|A_0\cdots A_{n-1})\\
=&\mu_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i_n},\quad \forall n; i_0,\cdots,i_n.
\end{aligned}
$$

- 若$\mu_i=1$, 则称$\{X_n\}$ 从i 出发.
- 初分布$\mu=\{\mu_i, i\in S\}$ & 发展机制$\mathbf{P}=(p_{ij})$

  完全确定全部有限为联合分布. 记为P_μ, P_i.
- 反过来, 若$P(X_0=i_0,\cdots,X_n=i_n)=\mu_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i_n}$,

  $\forall n, i_0,\cdots,i_n$, 则$(\star)$ 成立.

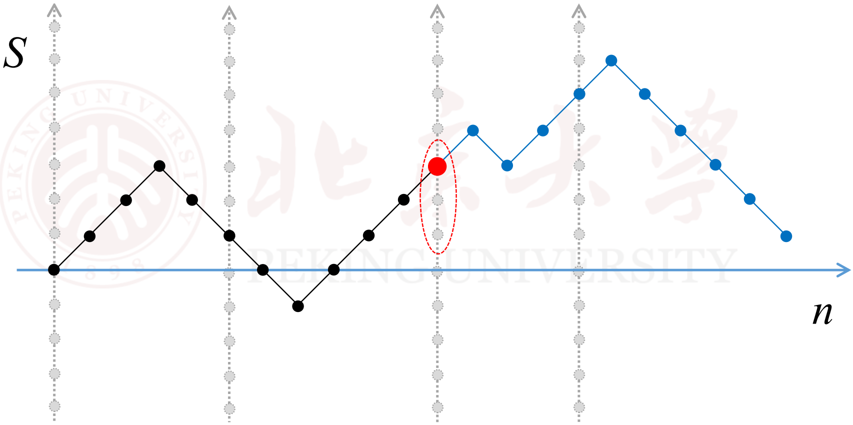
命题 (马氏性, 命题1.1.11)

取定$n \geqslant 1$, $i \in S$. 令$Y_m = X_{n+m}$, $\forall m \geqslant 0$. 那么, 在$\{X_n = i\}$ 的条件下, $\{Y_m\}$ 是从i 出发的以$\mathbf{P}$ 为转移矩阵的马氏链, 并且, 它与$\vec{Z} = (X_0, \cdots, X_{n-1})$ 相互独立.

注:

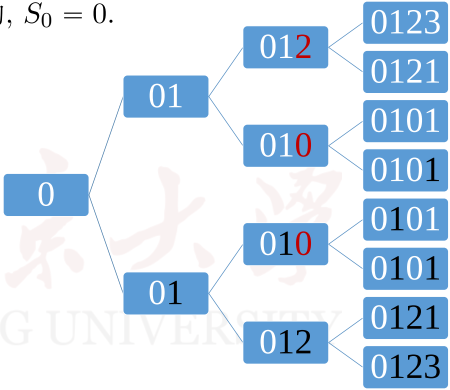
- 时刻n = 现在, 时间段$\{0, \cdots, n-1\}$ = 过去,
  时间段$\{n+1, n+2, \cdots\}$ = 将来.
- X_n: 现在(的状态), $\vec{Z}$: 过去(的经历), $\{Y_m\}$: 将来(的前途).
- 马氏性: 在知道现在的条件下, 过去与将来相互独立.

直观:



反例.

- 设$\{S_n\}$ 是一维简单随机游动, $S_0 = 0$.
- $n = 2$,
  $\hat{S} = \{0, 2\}$,
  $C_+ = \{X_0 = 0, X_1 = 1\}$,
  $C_- = \{X_0 = 0, X_1 = -1\}$,
- $j = 3$,
  $P(X_3 = 3 | X_2 \in \hat{S}) = \frac{1}{6}$.
  $P(X_3 = 3 | X_2 \in \hat{S}, C_+) = \frac{1}{4}$.
  $P(X_3 = 3 | X_2 \in \hat{S}, C_-) = 0$.
- $P(X_{n+1} = j | X_n \in \hat{S}, C_\pm) \neq P(X_{n+1} = j | X_n \in \hat{S})$.
- 对比§1.1 习题10.



命题 (马氏性, 命题1.1.11)

取定$n \geqslant 1$, $i \in S$. 令$Y_m = X_{n+m}$, $\forall m \geqslant 0$. 那么, 在$\{X_n = i\}$ 的条件下, $\{Y_m\}$ 是从i 出发的以$\mathbf{P}$ 为转移矩阵的马氏链, 并且, 它与$\vec{Z} = (X_0, \cdots, X_{n-1})$ 相互独立.

证明:

- $A = \{X_n = i\}$, $B = \{\vec{Y} = \vec{j}\}$, $C = \{\vec{Z} = \vec{i}\}$.
  其中$\vec{Y} = (Y_0, \cdots, Y_m)$, $\vec{j} = (j_0, \cdots, j_m)$. 所有有限维边缘.
- 需验证:
  (1) $P_A(B) = \mathbf{1}_{\{j_0=i\}} \times p_{j_0 j_1} \cdots p_{j_{m-1} j_m}$;
  (2) $P_A(BC) = P_A(B) P_A(C)$.
- 等价地, 需验证:

$$P_A(BC) = \mathbf{1}_{\{j_0=i\}} \times p_{j_0 j_1} \cdots p_{j_{m-1} j_m} P_A(C),$$

$$\text{或 } P_A(B|C) = \mathbf{1}_{\{j_0=i\}} \times p_{j_0 j_1} \cdots p_{j_{m-1} j_m}.$$

- $A=\{X_n=i\}$, $B=\{\vec{Y}=\vec{j}\}$, $C=\{\vec{Z}=\vec{i}\}$.
  其中$\vec{Y}=(Y_0,\cdots,Y_m)$, $\vec{j}=(j_0,\cdots,j_m)$. 所有有限维边缘.
- 往证

$$P_A(B|C)=\mathbf{1}_{\{j_0=i\}}\times p_{j_0j_1}\cdots p_{j_{m-1}j_m}.$$

- 当$j_0\neq i$ 时, LHS $=0=$ RHS. 当$j_0=i$ 时,

$$\text{LHS}=P_A(B|C)=P(B|A,C)=\frac{P(CAB)}{P(AC)}$$

$$=\frac{\mu_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i}\times p_{ij_1}\cdots p_{j_{m-1}j_m}}{\mu_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i}}$$

$$=p_{ij_1}\cdots p_{j_{m-1}j_m}=\text{RHS}.$$

n 步转移概率:

- $A=\{X_n=i\}$; $B=\{X_{n+m}=j_1\}$, $m\geqslant 1$;
  $C=\{(X_{n_1},\cdots,X_{n_r})=(i_1,\cdots,i_r)\}$,
  $0\leqslant n_1<\cdots<n_r<n$.
- 用$P(\cdot|A)$ 计算, $\{Y_m\}$ 与$\vec{Z}$ 独立, 因此B 与C 相互独立. 故

$$\begin{aligned}&P(X_{n+m}=j|X_n=i,X_{n_1}=i_1,\cdots,X_{n_r}=i_r)\\=&P(X_{n+m}=j|X_n=i)\triangleq p_{ij}^{(m)}.\end{aligned}$$

- 不依赖于n:

$$p_{ij}^{(m)}=\sum_{j_1,\cdots,j_{m-1}}P(C,X_{n+m}=j|X_m=i)$$

$$(\text{其中}, C=\{X_{n+1}=j_1,\cdots,X_{n+m-1}=j_{m-1}\})$$

$$=\sum_{j_1,\cdots,j_{m-1}}p_{ij_1}\cdots p_{j_{m-1}j}=(\mathbf{P}^m)_{ij}.$$

- n 步转移概率: $p_{ij}^{(n)}$.
- n 步转移矩阵: $(p_{ij}^{(n)})_{S\times S} = \mathbf{P}^n$.
- 对任意$0 < n_1 < \cdots < n_r$, $i_1, \cdots, i_r \in S$,

$$P(X_0 = i_0, X_{n_1} = i_1, X_{n_2} = i_2, \cdots, X_{n_r} = i_r)$$
$$= \mu_{i_0} p_{i_0 i_1}^{(n_1)} p_{i_1 i_2}^{(n_2 - n_1)} \cdots p_{i_{r-1} i_r}^{(n_r - n_{r-1})}.$$

- 命题1.1.13 (C-K等式) $p_{ij}^{(n+m)} = \sum_k p_{ik}^{(n)} p_{kj}^{(m)}$.
  - 代数的角度: $\mathbf{P}^{n+m} = \mathbf{P}^n \times \mathbf{P}^m$.
  - 概率的角度: 全概率公式.

$$P(X_{n+m} = j | X_0 = i) = \sum_k P(X_{n+m} = j, X_n = k | X_0 = i)$$
$$= \sum_k P(X_n = k | X_0 = i) P(X_{n+m} = j | X_n = k, X_0 = i).$$

四、马氏链的构造

- 思想: 用$U \sim U(0,1)$ 可以构造服从任意特定分布的随机变量. 具体地, 令$X = F^{-1}(U)$.
- 固定i. 取$f(i,\cdot)$ 实现$f(i,U) \sim \{p_{ij} : j \in S\}$.
- 对任意初分布μ, 取$g(\cdot)$ 实现$g(U) \sim \mu$.
- 取$U_0, U_1, U_2, \cdots$ i.i.d., $\sim U(0,1)$.
- 初始: 令$X_0 = g(U_0)$.
  迭代: $X_{n+1} = f(X_n, U_{n+1})$, $n = 0, 1, 2, \cdots$.

五、总结与补充

- 马氏链的定义、例子、建模. p_{ij}, $p_{ij}^{(n)}$.
- 有限维联合分布.

  $\{X_n\}$ 是以μ 为初分布、以$\mathbf{P}$ 为转移矩阵的马氏链:

$$P(X_0 = i_0, X_1 = i_1, \cdots, X_n = i_n) = \mu_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}.$$

- 随机试验、样本轨道、坐标过程、轨道分布.

$$\omega = i_0 i_1 i_1 \cdots, \quad X_n(\omega) = i_n, \quad \omega = (X_0(\omega), X_1(\omega), X_2(\omega), \cdots).$$

- 研究轨道的统计学性质.

§1.2 不变分布与可逆分布

一、不变分布

- 初分布μ & 转移机制$\mathbf{P} = (p_{ij})_{S\times S}$.

$$P_\mu(X_0 = i_0, X_1 = i_1, \cdots, X_n = i_n) = \mu_{i_0} p_{i_0 i_1} \cdots p_{i_{n-1} i_n}.$$

- $\mu_i = 1$ 时记P_i. 此时, $\mu_{i_0} = \mathbf{1}_{\{i_0 = i\}}$.
- 相应的期望分别记为E_μ, E_i.
- 关系式:

$$P_i(A) = P_\mu(A|X_0 = i), \quad P_\mu(A) = \sum_i \mu_i P_i(A).$$

- 本质: 按X_0 的值划分、全概率公式.

- X_n 的分布的递推式:

$$P_\mu(X_1 = j) = \sum_i P_\mu(X_0 = i, X_1 = j) = \sum_i \mu_i p_{ij} = (\mu\mathbf{P})_j.$$

- 定义1.2.1. 设$\pi = \{\pi_i : i \in S\}$ 为S 上的测度/分布.
  若π 满足如下不变方程:

$$\sum_i \pi_i p_{ij} = \pi_j, \ \forall j; \quad \text{即} \ \pi\mathbf{P} = \pi,$$

  则称π 为不变测度/不变分布 (invariant distribution).

- 平凡的不变测度: $\pi = \mathbf{0}$.
  非平凡的有限不变测度$\leftrightarrow$ 不变分布.

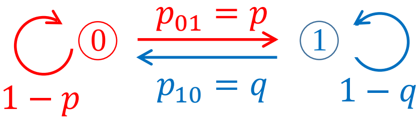
- 若$X_0 \sim \mu$, 则$X_n \sim \mu\mathbf{P}^n$;
  若$X_m \sim \mu$, 则$X_{m+n} \sim \mu\mathbf{P}^m$.
- 若π 为不变测度, 且$X_0 \sim \pi$. 则$\forall m, n$,

$$X_m \sim \pi \Rightarrow (X_0, X_1, \cdots, X_n) \overset{d}{=} (X_m, X_{m+1}, \cdots, X_{m+n}).$$

- 注: 满足$**$的过程称为平稳过程.
- 设S 有限. 则π 为$\mathbf{P}$ 的特征值为1 的左特征向量(不变方程$\pi = \pi\mathbf{P}$ 的解). 此时, 不变分布存在.
- 代数方法: Perron-Frobenius 定理;
  分析方法: §1.2 习题8;
  概率方法: §1.8.

例1.2.5. 两状态马氏链. $S=\{0,1\}$.

$$\mathbf{P}:\quad \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}.$$



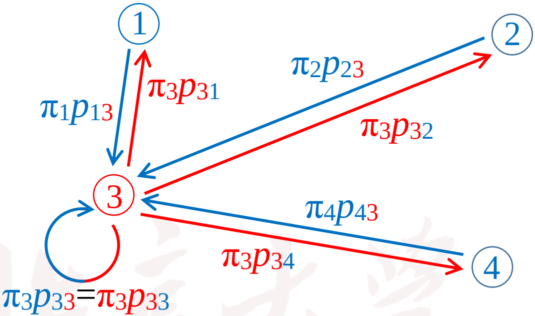
- $\pi\mathbf{P}=\pi$ 本质上只有1 个方程.

$$\left.\begin{array}{l} \pi_0 p+\pi_0(1-p)=\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\pi_0=\pi_1 q+\pi_0(1-p)} \\ \pi_0 p+\pi_1(1-q)=\underset{\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim\sim}{\pi_1=\pi_1 q+\pi_1(1-q)} \end{array}\right\} \Rightarrow \pi_0 p=\pi_1 q.$$

- 再补一个方程:

$$\left.\begin{array}{r} \pi_0 p=\pi_1 q \\ \pi_0+\pi_1=1 \end{array}\right\} \Rightarrow \pi_0=\frac{q}{p+q},\quad \pi_1=\frac{p}{p+q}.$$

流量.



- $\forall i$, $\sum_j \pi_j p_{ji} = \pi_i$,

  $\sum_{j\neq i} \pi_j p_{ji}$
  $= \pi_i \sum_{j\neq i} p_{ij}$
  $= \pi_i(1-p_{ii})$.

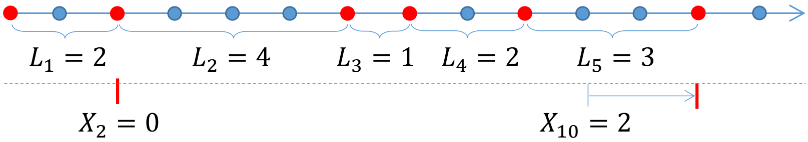
- 不变分布:
  对任意i, $\boxed{\text{总流入} = \text{总流出}}$.

- 进一步, $\sum_{i\in A}\sum_j \pi_j p_{ji} = \sum_{i\in A}\pi_i = \sum_{i\in A}\sum_j \pi_i p_{ij}$. 故

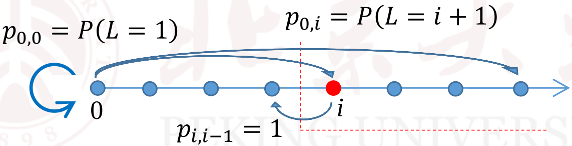
$$\underbrace{\sum_{j\notin A, i\in A} \pi_j p_{ji}} + \sum_{i,j\in A}\pi_j p_{ji} = \underline{\sum_{i\in A, j\notin A}\pi_i p_{ij}} + \sum_{i,j\in A}\pi_i p_{ij}.$$

- 注: 对不变分布, $\star\star\star = \star\star\star$. 对不变测度, 则未必.

例1.2.6. 更新过程. $L_1, L_2, \cdots$ i.i.d., 取正整数.



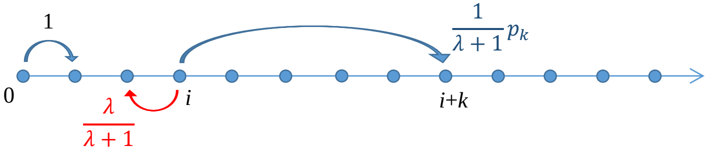
- X_n: 灯泡的余寿.



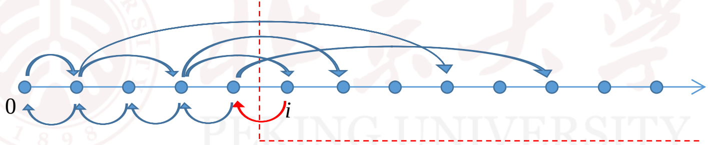
- 取$A = \{i, i+1, \cdots\}$. 推出$\pi_0 P(L \geqslant i+1) = \pi_i$.
- $\sum_i \pi_i = \pi_0(1 + P(L \geqslant 2) + P(L \geqslant 3) + \cdots) = \pi_0 EL$.
- 若$EL = \infty$, 则不变分布不存在.

  若$EL < \infty$, 则可验证$\{\frac{1}{EL} P(L \geqslant i+1) : i \geqslant 0\}$ 是不变分布.

例1.2.7. $S=\mathbb{Z}_+$. $1=\sum_{k=1}^{\infty} p_k < \sum_{k=1}^{\infty} kp_k < \lambda$. 求不变分布.



- 取$A_i=\{i,i+1,i+2,\cdots\}$.



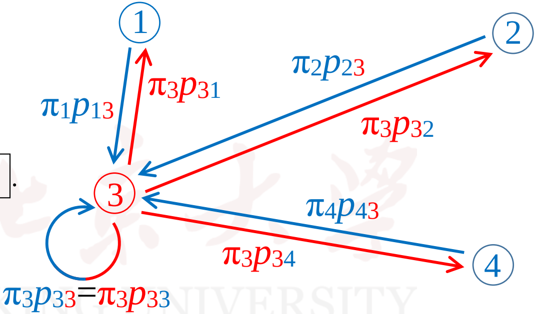
- 当$i=1$ 时: $\pi_1\frac{\lambda}{\lambda+1}=\pi_0$.
- 当$i\geqslant 2$ 时: $\pi_i\frac{\lambda}{\lambda+1}=\sum_{j=1}^{i-1}\pi_j\frac{f_{i-j}}{\lambda+1}$, 其中$f_r=p_r+p_{r+1}+\cdots$.
- $(1-\pi_0)\frac{\lambda}{\lambda+1}\times 1=\pi_0\times 1+(1-\pi_0)\frac{1}{\lambda+1}\times\sum_k kp_k$.
- 迭代得到其他π_i, 然后验证π 是不变分布.

二、可逆分布

- 不变分布: 对任意i,
  总流入 = 总流出.

- 充分条件: 流量对称.
  单项流入 = 单项流出.

- 细致平衡条件:
  $\pi_i p_{ij} = \pi_j p_{ji}, \forall i, j.$

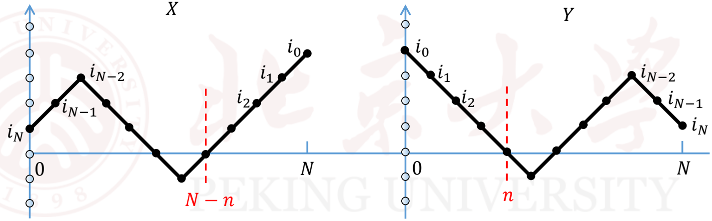


- 定义1.2.9. 满足细致平衡条件的非平凡的测度/分布称为配称测度/可逆分布. 此时, 称$\mathbf{P}$ 为可配称的/可逆的.

- 注: 可逆分布= 配称的不变分布.

逆过程.

- 假设π 是不变分布, $X_0 \sim \pi$.
- 固定$N \geqslant 2$. 令$Y_n = X_{N-n}$, $n = 0, 1, \cdots, N$.
  称$\{Y_n : 0 \leqslant n \leqslant N\}$ 为$\{X_n : 0 \leqslant n \leqslant N\}$ 的(时间倒)逆过程.



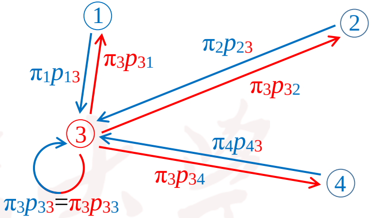
- 注: 记$m = N - n$. 视$Y_n = X_m$ 为现在的状态, 则
  逆过程的过去= 原过程的将来,
  逆过程的将来= 原过程的过去.
- 因此, $\{Y_n : 0 \leqslant n \leqslant N\}$ 是马氏链.

- 从直观推测逆过程的转移概率: 不妨假设$\pi_i > 0, \forall i$.
- 逆过程的转移概率$\tilde{p}_{ij}$ 应满足:

$$\pi_j p_{ji} = \pi_i \tilde{p}_{ij}, \quad \forall i, j.$$

- 令

$$\tilde{p}_{ij} = \frac{\pi_j p_{ji}}{\pi_i}, \ \forall i, j.$$



- $\{Y_n : 0 \leqslant n \leqslant N\}$ 是马氏链, 初分布为π, 转移概率为$\tilde{p}_{ij}$.

$$\text{其中}, \tilde{p}_{ij} = \frac{\pi_j p_{ji}}{\pi_i} \quad \text{满足 } \pi_j p_{ji} = \pi_i \tilde{p}_{ij}, \quad \forall i, j.$$

- 证: $\forall i_0, i_1, \cdots, i_N$,

$$\begin{aligned}
& P(Y_0 = i_0, Y_1 = i_1, \cdots, Y_{N-1} = i_{N-1}, Y_N = i_N) \\
=& P(X_0 = i_N, X_1 = i_{N-1}, \cdots, X_{N-1} = i_1, X_N = i_0) \\
=& \underline{\pi_{i_N} p_{i_N i_{N-1}}} p_{i_{N-1} i_{N-2}} \cdots p_{i_1 i_0} \\
=& \underline{\tilde{p}_{i_{N-1} i_N} \pi_{i_{N-1}}} p_{i_{N-1} i_{N-2}} \cdots p_{i_1 i_0} \\
=& \cdots = \tilde{p}_{i_{N-1} i_N} \tilde{p}_{i_{N-2} i_{N-1}} \cdots \tilde{p}_{i_0 i_1} \pi_{i_0}
\end{aligned}$$

- 若$\{X_n : 0 \leqslant n \leqslant N\}$ 与其逆过程同分布, 则称它为可逆的.
- 注: $\{X_n : 0 \leqslant n \leqslant N\}$ 可逆iff $\tilde{\mathbf{P}} = \mathbf{P}$ iff 细致平衡条件成立.
- 注: 若初分布为可逆分布, 也称$\{X_n : n \geqslant 0\}$ 是可逆.

§1.2 习题13. 设π 为不变分布. $\tilde{p}_{ij} = \pi_j p_{ji}/\pi_i$.

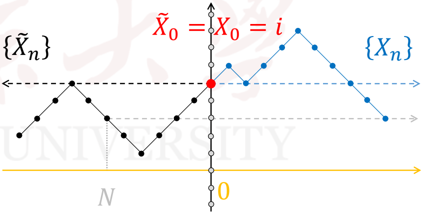
- Step 1. $X_0 = \tilde{X}_0 \sim \pi$,

  Step 2.1 在已知$\{X_0 = \tilde{X}_0 = i\}$ 的条件下, $\{X_n : n \geqslant 1\}$ 与$\{\tilde{X}_n : n \geqslant 1\}$ 相互独立.

  Step 2.2 $\{X_n\}$, $\{\tilde{X}_n\}$ 的转移矩阵分别为$\mathbf{P}$, $\tilde{\mathbf{P}}$.

- 令

$$Z_n = \begin{cases} X_n, & 若\ n \geqslant 0; \\ Y_{-n}, & 若\ n < 0. \end{cases}$$



- 结论: $\forall N \in \mathbb{Z}$, 令

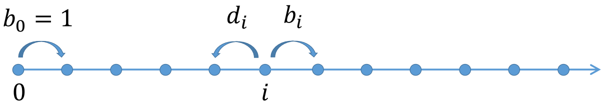
$$W_n = Z_{N+n}: \ n \geqslant 0.$$

则$\{W_n\}$ 是以π 为初分布, $\mathbf{P}$ 为转移矩阵的马氏链.

- 注: $\{Z_n : n \in \mathbb{Z}\}$ 是平稳过程.
- 注: $\tilde{Z}_n = Z_{-n}, \forall n$. 则$\{\tilde{Z}_n\}$ 是$\{Z_n\}$ 的逆过程.
- 注: $\{Z_n\}$ 可逆指$\{\tilde{Z}_n\} \stackrel{d}{=} \{Z_n\}$,
  iff π 为可逆分布.

计算可逆分布.

例1.2.12(生灭链). 在本例中假设$b_i, d_i > 0$, $i \geqslant 1$.



- $\pi_0 b_0 = \pi_1 d_1 \Rightarrow \pi_1 = \pi_0 \frac{b_0}{d_1}$.
- $\pi_1 b_1 = \pi_2 d_2 \Rightarrow \pi_2 = \pi_1 \frac{b_1}{d_2} = \pi_0 \frac{b_0 b_1}{d_1 d_2}$.
- $\pi_{i-1} b_{i-1} = \pi_i d_i \Rightarrow \pi_i = \pi_{i-1} \frac{b_{i-1}}{d_i} = \cdots = \pi_0 \frac{b_0 \cdots b_{i-1}}{d_1 \cdots d_i}$.
- 归一化:

  若$C := 1 + \frac{b_0}{d_1} + \frac{b_0 b_1}{d_1 d_2} + \cdots = \infty$, 则没有可逆分布/不变分布.

  若$C < \infty$, 则π 是(唯一的)可逆分布/不变分布.

$$\pi_0 = \frac{1}{C}, \quad \pi_i = \frac{1}{C} \times \frac{b_0 \cdots b_{i-1}}{d_1 \cdots d_i}, \quad i \geqslant 1.$$

配称测度/可逆分布π 的优越性:

- 容易计算:

  (1) 固定o.

  $\forall i \neq o$, 找$n \geqslant 1$ 以及$i_0 := o, i_1, \cdots, i_n := i$, 使得$p_{i_0 i_1} \cdots p_{i_{n-1} i_n} > 0$, 并令

$$\pi_i = \pi_o \frac{p_{i_0 i_1} \cdots p_{i_{n-1} i_n}}{p_{i_1 i_0} \cdots p_{i_n i_{n-1}}}.$$

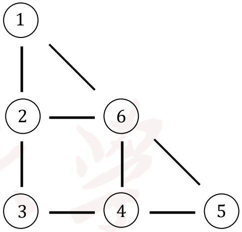
  (2) 验证细致平衡条件: $\pi_i p_{ij} = \pi_j p_{ji}$, $\forall i, j$.

  (3) 将π 归一化.

- 具有继承性: $\forall D \subseteq S$, π 仍是$\tilde{\mathbf{P}} = (\tilde{p}_{ij})_{D\times D}$ 的配称测度.

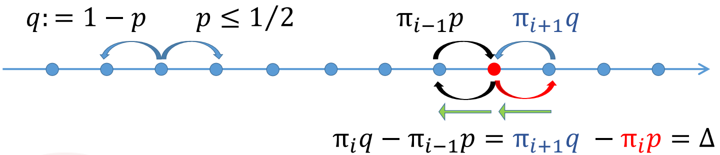
$$\begin{cases} \tilde{p}_{ij} = p_{ij}, & \forall\ i, j \in D,\ j \neq i, \\ \tilde{p}_{ii} = p_{ii} + \sum_{j \notin D} p_{ij}, & \forall i \in D. \end{cases}$$

例1.2.13. 有限连通图上的随机游动.



- §1.9 习题2.
- $w_{ij} = \pi_i p_{ij} = \pi_i / d_i, \ \forall j \sim i.$
- 细致平衡条件:
$$w_{ij} = w_{ji} \Rightarrow \frac{\pi_i}{d_i} = \frac{\pi_j}{d_j}, \forall j \sim i.$$
- 图连通, 故$\pi_i = C d_i, \ \forall i.$
- 归一化: $C = 1 / \sum_i d_i.$

例1.2.14. 一维紧邻随机游动. 不妨设$0 < p \leqslant 1/2$.



- 可逆分布不存在:

  否则$\Delta := \pi_i p - \pi_{i+1} q = 0$, 于是$\pi_i \geqslant \pi_{i+1}$, 矛盾!

- 不变分布不存在.

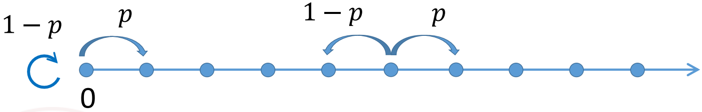
  取$A = \{i+1, i+2, \cdots\}$, 则$\Delta = 0$, 转化为可逆分布.

- 配称测度存在且(在允许差常数倍的意义下)唯一:

$$\pi_i = \pi_0 \frac{p^i}{q^i}, \quad \forall i; \quad \left(\pi_{-i} = \pi_0 \frac{p^{-i}}{q^{-i}}, \ i \geqslant 1\right).$$

- 若$p \neq 1/2$, 则不变测度不唯一. 另有$\pi_i \equiv c$. (注: $\Delta \neq 0$).
- 若$p = 1/2$, 则不变测度唯一. (§1.8 补充知识).

- 带粘滞边界的随机游动.



- 配称测度存在且唯一:

$$\pi_i = \pi_0 \frac{p^i}{q^i}, \quad \forall i \geqslant 0.$$

- 若$p < 1/2$, 则可归一化为可逆分布.
- 若$p \geqslant 1/2$, 则不变分布不存在.

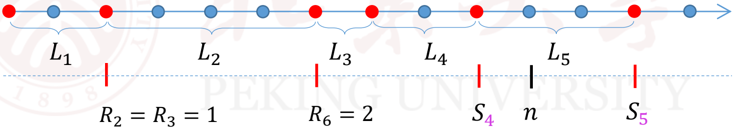
  此时, π 为唯一的不变测度/配称测度.

三、访问频率

- 设$L_1, L_2, \cdots$ i.i.d., 取非负整数, 且$P(L_1 = 0) < 1$. 令

$$S_0 = 0, \quad S_r := L_1 + \cdots + L_r, \; r \geqslant 1,$$

$$R_n := \max\{r \geqslant 0 : S_r \leqslant n\}.$$

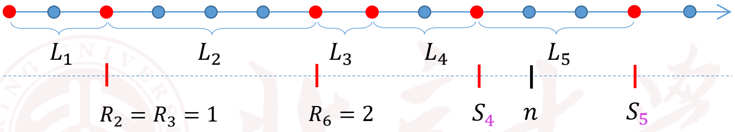


- 更新定理:

$$P\left(\lim_{n\to\infty} \frac{R_n}{n} = \frac{1}{EL_1}\right) = 1.$$

定理 (更新定理, 定理1.2.15)

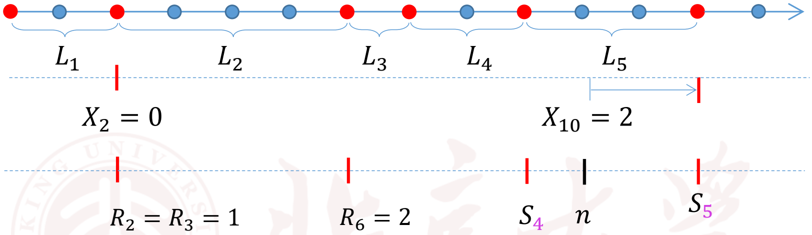
$$P\left(\lim_{n\to\infty}\frac{R_n}{n}=\frac{1}{EL_1}\right)=1.$$



- $R_n = r$ iff $S_r \leqslant n < S_{r+1}$.
- $R_n \nearrow R = \infty$, a.s.: $R < \infty$ iff $\exists r$ 使得$L_r = \infty$.
- SLLN: $S_r/r \to a := EL_1$, a.s..
- $P(\Omega_1) = P(\Omega_2) = 1$.
- $\forall \omega \in \Omega_1\Omega_2$, $r_n := R_n(\omega) \to \infty$, $a_r := S_r(\omega) \approx ar$,

  且$a_{r_n} \leqslant n < a_{r_n+1}$. 故$n/r_n \to a$.

例1.2.16. 更新过程. $L_1, L_2, \cdots$ i.i.d., 取正整数.



- 若$EL_1 = \infty$, 则不变分布不存在.

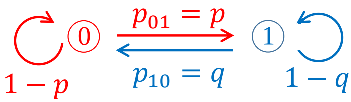
  若$EL_1 < \infty$, 则不变分布为$\{\frac{1}{EL_1} P(L \geqslant i+1) : i \geqslant 0\}$.

- 访问状态0 的频率: 由更新定理,

$$\frac{1}{n+1}|\{m \leqslant n : X_m = 0\}| = \frac{R_{n+1}}{n+1} \to \frac{1}{EL_1} = \pi_0.$$

例1.2.17. 两状态马氏链. $\pi_0 = \frac{q}{p+q}$.

- 设$X_0 = 0$. 轨道:



$$\underbrace{0\cdots0}_{\xi_1}\ \underbrace{1\cdots1}_{\eta_1}\ \underbrace{0\cdots0}_{\xi_2}\ \underbrace{1\cdots1}_{\eta_2}\cdots$$

- 相互独立. $\xi = \xi_1$ 与$\eta = \eta_1$ 的分布:

$$P(\xi = m) = (1-p)^{m-1}p, \quad P(\eta = m) = (1-q)^{m-1}q, \quad m \geqslant 1.$$

- 取$L = \xi + \eta$; L_r, $r \geqslant 1$. 则

$$V_0(n) := |\{m \leqslant n : X_n = 0\}| = \xi_1 + \cdots + \xi_r + \zeta_n.$$

其中, $r = R_n$, $0 \leqslant \zeta_n \leqslant \xi_{r+1}$.

- 访问状态0 的频率: 由更新定理,

$$\frac{1}{n+1}V_0(n) \to (E\xi + 0) \times \frac{1}{EL} = \frac{1/p}{1/p + 1/q} = \frac{q}{p+q} = \pi_0.$$

四、访问概率的收敛性

例1.2.18. 两状态马氏链. $0 < p, q < 1$.
研究$P(X_n = 0)$ 的极限.

$$\begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & \quad 1 \end{array} \\ \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix} \end{array}$$

- 若$X_0 \sim \mu$, 则$X_n \sim \mu \mathbf{P}^n$.
  归结于研究$\hat{\mathbf{P}} = \lim_{n\to\infty} \mathbf{P}^n$.
- 相似变换: $\lambda := 1 - p - q \in (-1, 1)$,

$$\mathbf{P} = \mathbf{A}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & \boxed{\lambda} \end{pmatrix} \mathbf{A}, \quad \text{其中} \mathbf{A} = \begin{pmatrix} \frac{q}{p+q} & \frac{p}{p+q} \\ \frac{1}{p+q} & -\frac{1}{p+q} \end{pmatrix}.$$

- $\hat{\mathbf{P}} = \begin{pmatrix} 1 & p \\ 1 & -q \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \boxed{0} \end{pmatrix} \begin{pmatrix} \pi_0 & \pi_1 \\ * & * \end{pmatrix} = \begin{pmatrix} \pi_0 & \pi_1 \\ \pi_0 & \pi_1 \end{pmatrix}.$

- 结论1:

$$p_{ij}^{(n)} \to \pi_j, \quad \forall i, j;$$

$$P_\mu(X_n = j) = \mu_0 p_{0j}^{(n)} + \mu_1 p_{1j}^{(n)} \to \pi_j, \quad \forall \mu, j.$$

- 结论2:

$$\mathbf{P}^n - \hat{\mathbf{P}} = \mathbf{A}^{-1} \begin{pmatrix} 0 & 0 \\ 0 & \lambda^n \end{pmatrix} \mathbf{A} = \lambda^n \begin{pmatrix} \frac{p}{p+q} & -\frac{p}{p+q} \\ -\frac{q}{p+q} & \frac{q}{p+q} \end{pmatrix},$$

因此 $p_{ij}^{(n)} - \pi_j = O(\lambda^n)$, 其中, $|\lambda| < 1$.

§1.1 习题5. 投球成功的概率取决于前两次的投球成绩.

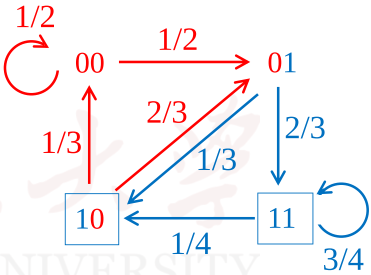
(2) 求第n 次投球成功的概率p_n 的极限.

- 第n 次: $\xi_n = 1$ (成功) 或0.

  $X_n = \xi_n \xi_{n+1}$.

- $\pi = (\frac{1}{8}, \frac{3}{16}, \frac{3}{16}, \frac{1}{2})$:

$$\begin{cases} \frac{1}{2}\pi_{00} = \frac{1}{3}\pi_{10}, \\ \frac{2}{3}\pi_{01} = \frac{1}{4}\pi_{11}, \\ \pi_{10} = \pi_{01}. \end{cases}$$



- 验证: $p_{ij}^{(n)} \to \pi_j, \forall i, j$. 于是,

$$p_n = P(\xi_n = 1) = P(X_n = 10) + P(X_n = 11) \to \pi_{00} + \pi_{11} \to \frac{11}{16}.$$

五、总结与补充

- 不变分布、可逆分布的定义与计算; 逆过程.
- 更新定理、i 的访问频率$\to \pi_i$.
- $P(X_n = i) \to \pi_i$.
- 不变分布π 的存在(唯一)性.
  正常返(不可约). §1.8 (§1.3).
- 访问i 的频率$\to \pi_i$.
  遍历定理. §1.8
- $\mu \mathbf{P}^n \to \pi$.
  强遍历定理. §1.9
- 铺垫工作: §1.3 互通性, §1.4 强马氏性, §1.5 ~ 1.7 常返性.

§1.3 状态的分类

- 定义1.3.1. 可达, $i \to j$: $P_i(\exists n \geqslant 0$ 使得$X_n = j) > 0$.
- 或者$j = i$,

  或者$\exists n \geqslant 1, i_0, \cdots, i_n$,

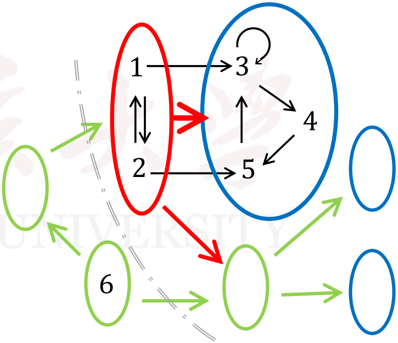
  $(i_0 = i, i_n = j)$,

  使得$p_{i_0 i_1} \cdots p_{i_{n-1} i_n} > 0$.
- 存在$n \geqslant 0$ 使得$p_{ij}^{(n)} > 0$.

  证明: $P_i(X_n = j)$

  $\leqslant P_i(\exists m \geqslant 0$ 使得$X_m = j)$

  $\leqslant \sum_{m=0}^{\infty} P_i(X_m = j)$.



- 定义1.3.3. 互通$i \leftrightarrow j$: $i \to j$ 且$j \to i$.

  若所有i, j 都互通, 则称$\mathbf{P}$ (S, 马氏链)不可约, 否则称可约.

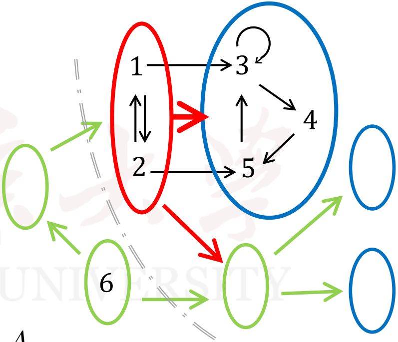
- 互通类: $[i] = \{j : j \leftrightarrow i\}$.

- 定义1.3.4. 若

$$p_{ij} = 0, \quad \forall i \in A, \ \forall j \notin A,$$

  则称A 为闭集/闭的.

  闭的互通类称为闭类.



- 命题1.3.5. 若A 为闭集, 则

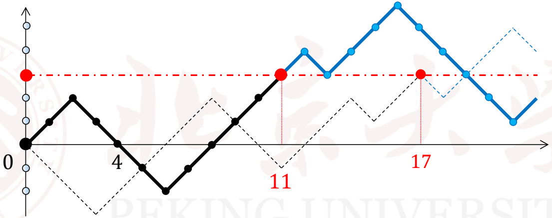
$$P_i(X_n \in A, \forall n \geqslant 0) = 1, \quad \forall i \in A.$$

- 命题1.3.6. 假设A 为互通类, 且不是闭集, 则

$$P_i(\exists n \geqslant 0 \text{ 使得} X_n \notin A) > 0, \quad \forall i \in A.$$

一、首达时与首入时

- 样本轨道: $\omega = \Big(X_0(\omega), X_1(\omega), X_2(\omega), \cdots\Big) = \vec{X}(\omega)$.
- 首达(中)时(hitting time) τ_i: 例, $\tau_3(\omega) = 11$, $\tau_3(\hat{\omega}) = 17$.



- τ_i 是随机变量. 约定$\inf \emptyset = \infty$.

$$\tau_i(\omega) := \inf\{n \geqslant 0 : X_n(\omega) = i\}.$$

- τ_i 是$\{X_n\}$ 的函数, 也记为$\tau_i^{(X)}$.

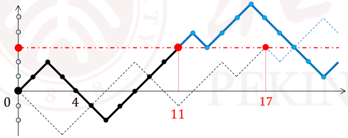
$$\tau_i(\vec{x}) := \inf\{n \geqslant 0 : x_n = i\}, \quad \tau_i = \tau_i(\vec{X}).$$

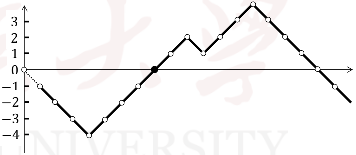
- 首入时σ_i 是随机变量，是$\{X_n\}$ 的函数，也记为$\sigma_i^{(X)}$.

$$\sigma_i(\omega) := \inf\{n \geqslant 1 : X_n(\omega) = i\}.$$

- 注：σ_i 常用于$X_0 = i$ 的情形． 否则，常用τ_i.

  例，$\tau_0(\omega) = 0$, $\sigma_0(\omega) = 4$.





- σ_i 与τ_i 的联系：例1.4.2. $\sigma_i^{(X)} = 1 + \tau_i^{(Y)}$.
- 类似地，τ_D, $\tau_D^{(X)}$, σ_D, $\tau_D^{(Y)}$.

二、一维简单随机游动的首达时

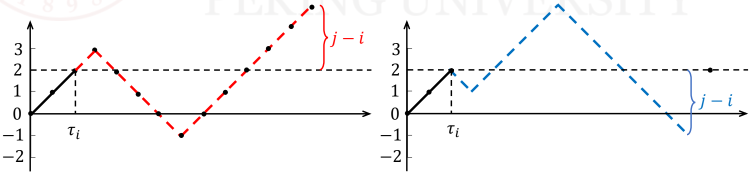
- $\xi_1, \xi_2, \cdots$ i.i.d., $P(\xi_1 = 1) = P(\xi_1 = -1) = \frac{1}{2}$.

$$S_0 = 0, \quad S_n = \xi_1 + \cdots + \xi_n.$$

命题 (反射原理, 命题1.4.3)

$$P_0(\tau_i < n, S_n = i + j) = P_0(\tau_i < n, S_n = i - j).$$

- $i = 0$, ✓. 不妨设$1 \leqslant i < n$.
- 前n 步轨道: $\{\tau_i < n, S_n = i + j\} \leftrightarrow \{\tau_i < n, S_n = i - j\}$.



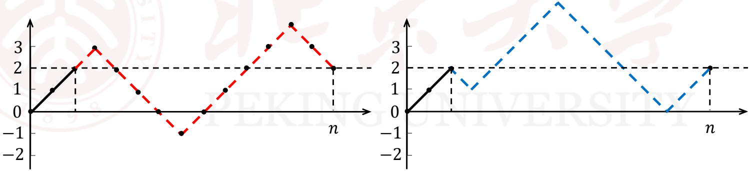
- n 步轨道的数目相同, 故概率相同.

命题 (命题1.4.4)

$$P_0(\tau_i = n) = \frac{i}{n} P_0(S_n = i), \quad \forall i, n \geqslant 1.$$

- 不妨设$n+i$ 为偶数. $\tau_i = n$ iff $S_n = i$, $\underset{\sim}{S_1, \cdots, S_{n-1} \neq i}$.
- $\{S_n = i\} - \{\tau_i = n\} = \{\underset{\sim}{\tau_i < n}, S_n = i\}$.

$$\{\underset{\sim}{\tau_i < n}, S_{n-1} = i+1; S_n = i\} \leftrightarrow \{\underset{\sim}{\tau_i < n}, S_{n-1} = i-1; S_n = i\}.$$



- $P(A) = 2P(S_{n-1} = i+1, S_n = i) = P(S_{n-1} = i+1)$.
- $P(\tau_i = n) = P(S_n = i) - \star\star$,

$$\star\star = C_{n-1}^{(n+i)/2} \frac{1}{2^{n-1}} = \frac{n-i}{n} P(S_n = i).$$

推论1.4.5. $P_0(\tau_1 > 2n-1) = P_0(S_{2n}=0)$.

- 证: τ_1 是奇数, $\{\tau_1 > 2n-1\}^c = \{\tau_1 \leqslant 2n-1\} = \{\tau_1 < 2n\}$.
- $\{\tau_1 < 2n, S_{2n} = 1+j\} \leftrightarrow \{\tau_1 < 2n, S_{2n} = 1-j\}$.
- S_{2n} 是偶数, 故

$$P(\tau_1 < 2n) = 2P_0(S_{2n} \geqslant 2) = P_0(S_{2n} \neq 0).$$

推论1.4.6. $P_0(\tau_1 < \infty) = P_0(\sigma_0 < \infty) = 1$, $E_0\tau_1 = E_0\sigma_1 = \infty$.

- 证: $P_0(\tau_1 > 2n-1) = P_0(S_{2n}=0) = C_{2n}^n 2^{-n} \approx \frac{1}{\sqrt{\pi n}}$.
- 因为$S_0 = 0$, 所以$\sigma_0 \overset{d}{=} 1+\tau_1$.
- 注: 零常返, §1.5, §1.8.

命题 (命题1.4.8. 反正弦律)

$$\lim_{n\to\infty} P_0\big(\epsilon_n \leqslant \delta n\big) = \frac{2}{\pi}\arcsin\sqrt{\delta}, \quad \forall \delta \in (0,1).$$

其中, $\epsilon_n := \max\{m \leqslant n : S_m = 0, S_{m+1} \neq 0, \cdots, S_n \neq 0\}$.

- 等价地, $\lim_{n\to\infty} P_0(S_r \neq 0, \forall\ \delta n < r \leqslant n) = \frac{2}{\pi}\arcsin\sqrt{\delta}$.
- 令$m = [\delta n]$. 则$\star = \sum_i P(S_m = i)P_i(\tau_0 > n-m)$.
- $1 - \star = \sum_i P(S_m = i)P_0(\tau_i \leqslant n-m)$.
- $\forall i \geqslant 1$,

$$\{\tau_i \leqslant n-m\} = \{\underset{\sim\sim\sim}{\tau_i \leqslant n-m}, S_{n-m} = i\} + A + B,$$

$$A := \{\underset{\sim\sim\sim}{\tau_i < n-m}, S_{n-m} > i\} \leftrightarrow B := \{\tau_i < n-m, S_{n-m} < i\}.$$

- $P_0(\tau_i \leqslant n-m) = P(S_{n-m} = i) + 2P(S_{n-m} > |i|)$.

- 目标: $\lim_{n\to\infty} P_0(S_r \neq 0, \forall\ \delta n < r \leqslant n) = \frac{2}{\pi}\arcsin\sqrt{\delta}$.
- 已证: 令$m = [\delta n]$. 则$1 - \star = \sum_i P(S_m = i)P_0(\tau_i \leqslant n - m)$,

  $P_0(\tau_i \leqslant n - m) = P(S_{n-m} = i) + 2P(S_{n-m} > |i|)$.
- 取与$\{S_n\}$ 独立的SRW $\{T_n\}$:

$$P_0(\tau_i \leqslant n - m) = P(T_{n-m} = i) + P(|T_{n-m}| > |i|).$$

- $1 - \star \approx \sum_i P(S_m = i)P(|T_{n-m}| > |i|) = P(|T_{n-m}| > |S_m|)$.
- $\star = P(|T_{n-m}| \leqslant |S_m|) \to \frac{2}{\pi}\arcsin\sqrt{\delta}$:

$$P\left(\sqrt{n-m}\cdot\left|\frac{T_{n-m}}{\sqrt{n-m}}\right| \leqslant \sqrt{m}\cdot\left|\frac{S_m}{\sqrt{m}}\right|\right) \approx P\left(\left|\frac{W}{Z}\right| \leqslant \frac{\sqrt{\delta}}{\sqrt{1-\delta}}\right).$$

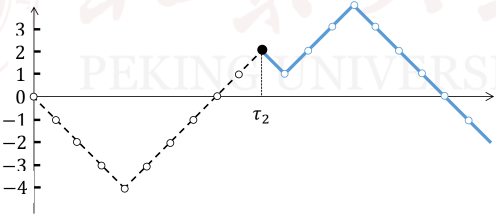
- 注: 布朗运动也有, §3.4.

三、强马氏性

- 取定$i \in S$, 记$\tau = \tau_i$.
- 在事件$\{\tau < \infty\}$ 上, 令

$$(Y_0, Y_1, Y_2, \cdots) = (X_\tau, X_{\tau+1}, X_{\tau+2}, \cdots),$$

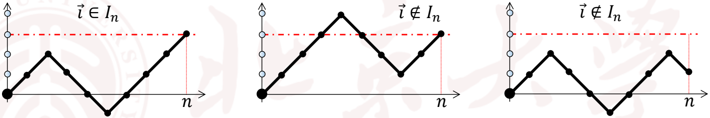
$$\vec{Z} = (X_0, \cdots, X_\tau).$$



- 以下采用$P_{\{\tau<\infty\}}(\cdot)$. 可认为$\{Y_n\}$ 与$\vec{Z}$ 的定义已经完整.

- $\vec{Z}=(X_0,\cdots,X_\tau)$: 离散型, 值域$I\subseteq S^1\bigcup S^2\bigcup\cdots$.
- $I:=\bigcup_{n=0}^{\infty} I_n$, 记 $\vec{i}=(i_0,i_1,\cdots,i_n)$,

$$I_n:=\left\{\vec{i}\in S^{n+1}: i_0,\cdots,i_{n-1}\neq i \text{ 且 } i_n=i\right\}.$$



- 记$\vec{X}^{(n)}:=(X_0,\cdots,X_n)$. $\forall\vec{i}\in I_n$,

$$\{\vec{Z}=\vec{i}\}=\{\vec{X}^{(n)}=\vec{i}\}$$

$$\Rightarrow P(\vec{Z}=\vec{i})=\mu_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i_n}.$$

- 注: 可能仍然$\exists\vec{i}\in I$ 使得$P(\vec{Z}=\vec{i})=0$, 但没必要再筛选.

- $\{\vec{Z}=\vec{i}\}=\{\vec{X}^{(n)}=\vec{i}\},\quad \forall \vec{i}\in I_n,$
- $\{\tau=n\}=\sum_{\vec{i}\in I_n}\star\star$:

$$\star\star\subseteq\{\tau=n\},\quad \forall \vec{i}\in I_n;\qquad \tau=n\Rightarrow \vec{X}^{(n)}\in I_n.$$

- 因此

$$\begin{aligned}P(\tau<\infty)&=\sum_{n=0}^{\infty}P(\tau=n)\\&=\sum_{n=0}^{\infty}\sum_{\vec{i}\in I_n}P\Big(\star\star\Big)=\sum_{n=0}^{\infty}\sum_{\vec{i}\in I_n}P\Big(\star\star\Big)\\&=\sum_{n=0}^{\infty}\sum_{\vec{i}\in I_n}\mu_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i_n}.\end{aligned}$$

- $\forall n\geqslant 0,\ \vec{i}\in I_n,$

$$P_{\{\tau<\infty\}}(\vec{Z}=\vec{i})=\mu_{i_0}p_{i_0i_1}\cdots p_{i_{n-1}i_n}/\star\star.$$

命题 (命题1.4.10)

在$\{\tau_i < \infty\}$ (或$\{\tau_i = n\}$)发生的条件下,

(1) $\{Y_m\}$ 是从i 出发的马氏链; (2) $\{Y_m\}$ 与$\vec{Z}$ 相互独立.

- 记$\tau_i = \tau$. 假设$\{\tau = n\}$ 发生. 这蕴含着$\vec{Z} = \vec{X}^{(n)} \in I_n$.
- $\vec{i} := (i_0, \cdots, i_n) \in I_n$; $\vec{j} := (j_0, \cdots, j_m)$,

$$B := \{(Y_0, \cdots, Y_m) = \vec{j}\} = \{(X_\tau, \cdots, X_{\tau+m}) = \vec{j}\},$$

$$C := \{\vec{Z} = \vec{i}\} = \{(X_0, \cdots, X_\tau) = \vec{i}\}.$$

- 只需验证:

  $\forall \vec{i} \in I_n$; $\forall m \geqslant 0, \forall \vec{j} \in S^{m+1}$,

$$P_{\{\tau=n\}}(BC) = 1_{\{j_0=i\}} p_{j_0 j_1} \cdots p_{j_{m-1} j_m} \times P_{\{\tau=n\}}(C).$$

- $B=\{(X_\tau,\cdots,X_{\tau+m})=\vec{j}\}$, $C=\{\vec{Z}=\vec{i}\}$. 其中, $\vec{i}\in I_n$.
- 令$\hat{B}:=\{(X_n,\cdots,X_{n+m})=\vec{j}\}$, $\hat{C}:=\{\vec{X}^{(n)}=\vec{i}\}$.
- 因为 $\vec{i}\in I_n$, 所以

$$C=\hat{C}\subseteq D,\quad D:=\{\tau=n\}.$$

- $BD=\hat{B}D$. (注: $B\neq\hat{B}$.)
- 综上,

$$BC=BDC=\hat{B}D\hat{C}=\hat{B}\hat{C}.$$

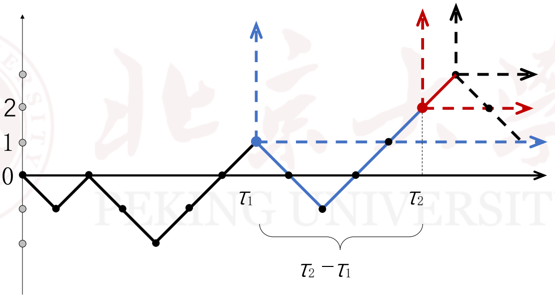
注: $\vec{i}\notin I_n$ 时, 不可以推出$BC=\hat{B}\hat{C}$.

- $B=\{(X_\tau,\cdots,X_{\tau+m})=\vec{j}\}$, $C=\{\vec{Z}=\vec{i}\}$. 其中, $\vec{i}\in I_n$.
- $\hat{B}:=\{(X_n,\cdots,X_{n+m})=\vec{j}\}$, $\hat{C}:=\{\vec{X}^{(n)}=\vec{i}\}$.
- $C=\hat{C}\subseteq\{\tau=n\}$; $BC=\hat{B}\hat{C}$.
- 于是

$$P(BC)=P(\hat{B}\hat{C})=1_{\{j_0=i\}}p_{j_0j_1}\cdots p_{j_{m-1}j_m}\times P(\hat{C})=\star\times P(C).$$

- 两边同时除以$P(\tau=n)$ 即可.
- 在$\{\tau<\infty\}$ 的条件下, 只需验证$\forall\vec{i}\in I=\sum_n I_n$.
  此时, $\exists! n$ 使得$\vec{i}\in I_n$. 两边同时除以$P(\tau<\infty)$ 即可.

- 推论1.4.12. 若$P(\tau_i < \infty) = 1$, 则$\{Y_m\}$ 是从i 出发的马氏链, 且与$\vec{Z}$ 相互独立.
- 推论1.4.13. 将τ_i 改为σ_i, 则命题1.4.10 与推论1.4.12 仍成立.
- 例1.4.15. SRW. $\tau_i = \tau_1^{(1)} + \cdots + \tau_1^{(i)}$.

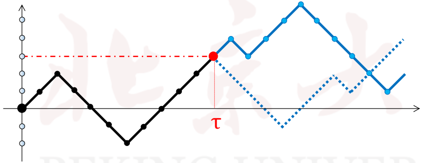


- 另证$E_0\tau_1 = \infty$:

$$E_0\tau_1 = \frac{1}{2} \times 1 + \frac{1}{2} \times (1 + E_0\tau_2) = 1 + \frac{1}{2}E_0\tau_2 = 1 + E_0\tau_1.$$

四、总结与补充

- τ_i 与σ_i.
- SRW: τ_i 的分布; 反射原理; $P(\tau_i < \infty) = 1$, $E_0\tau_1 = \infty$.
- 强马氏性: 设$P(\tau_i < \infty) = 1$. 则轨道可拼接:

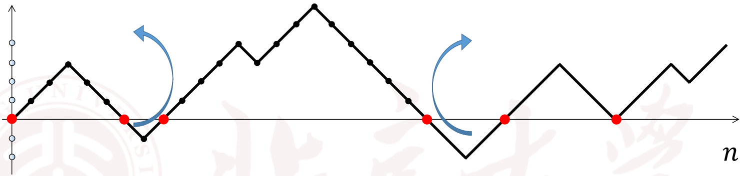


- 对第r 次访问i 的时刻, 强马氏性都成立.
- 对停时, 强马氏性成立.
- 注1.4.14: 马氏性: 将固定的时刻n 视为现在.

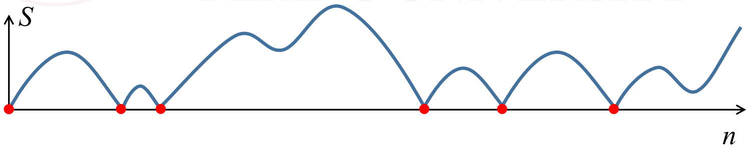
  强马氏性: 可将τ_i, σ_i 等随机时刻视为现在.

§1.5 常返性

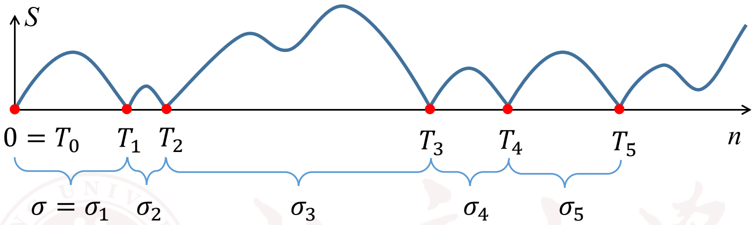
- 可达$j \to i$: $P_j(\tau_i < \infty) > 0$. 访问i 多少次? 所有访问时间?



- 轨道示意图:



一、回访时间



- 固定i. 递归定义$T_r := T_{i,r}$, $\sigma_r := \sigma_{i,r}$. 令$T_0 := 0$. $\forall r \geqslant 1$, 若$T_{r-1} < \infty$, 则

$$T_r := \inf\{n \geqslant T_{r-1} + 1 : X_n = i\}, \quad \sigma_r := T_r - T_{r-1};$$

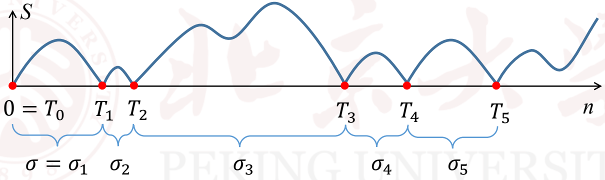
若$T_{r-1} = \infty$, 则$T_r := \infty$, $\sigma_r := \infty$.

- 设$X_0 = i$. 此时, $T_{i,1} = \sigma_i$. 注: 若$X_0 \neq i$, $T_{i,1} = \tau_i$.
- 关键量: $\sigma_i := \inf\{n \geqslant 1 : X_n = i\}$.

命题 (命题1.5.1)

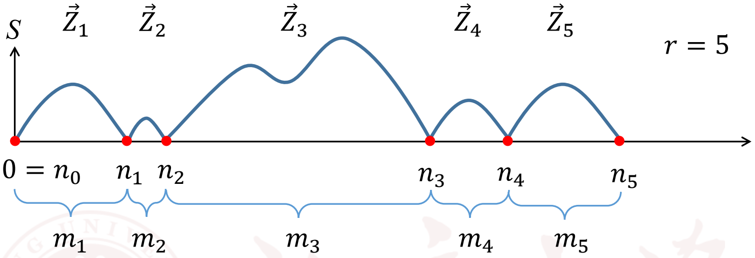
(1) $\forall r \geqslant 1;\ m_1, \cdots, m_r \geqslant 1$,

$$P_i(\sigma_1 = m_1, \sigma_2 = m_2, \cdots, \sigma_r = m_r) = \prod_{s=1}^{r} P_i(\sigma = m_s).$$



- 固定$n \geqslant 1$. 记$\vec{i} := (i_0, i_1, \cdots, i_n)$, 令

$$J_n := \left\{\vec{i} \in S^{n+1} : i_0 = i_n = i,\ 且\ i_r \neq i,\ r = 1, \cdots, n-1\right\}.$$



- 证：令$n_s=\sum_{t=1}^s m_t$; $\vec{Z}_s=(X_{n_{s-1}},\cdots,X_{n_s})$. 则

$$\text{LHS}=\sum_{\vec{i}_1\in J_{m_1},\cdots,\vec{i}_r\in J_{m_r}} P_i\left(\vec{Z}_1=\vec{i}_1,\cdots,\vec{Z}_r=\vec{i}_r\right)$$

$$=\sum_{\vec{i}_1\in J_{m_1}}\cdots\sum_{\vec{i}_r\in J_{m_r}} P_i\left(\vec{X}^{(m_1)}=\vec{i}_1\right)\cdots P_i(\vec{X}^{(m_r)}=\vec{i}_r)$$

$$=\prod_{s=1}^r\sum_{\vec{i}_s\in J_{m_s}} P_i\left(\vec{X}^{(m_s)}=\vec{i}_s\right)=\text{RHS}.$$

- $P_i(\sigma_1 = m_1, \sigma_2 = m_2, \cdots, \sigma_r = m_r) = \prod_{s=1}^r P_i(\sigma = m_s)$.
- 命题1.5.1.

  (2). 进一步, 若$\rho_i := P_i(\sigma < \infty) = 1$, 则$\sigma_1, \sigma_2, \cdots$ i.i.d..
- 证: $\star$ 表明$\sigma_1, \sigma_2, \cdots$ 取正整数值, 从而✓.
- 推论1.5.3. $P_i(T_r < \infty) = {\rho_i}^r$.
- 证: 用归纳法. 按定义$\sigma_r = \infty \Rightarrow \sigma_{r+1} = \infty$. 故

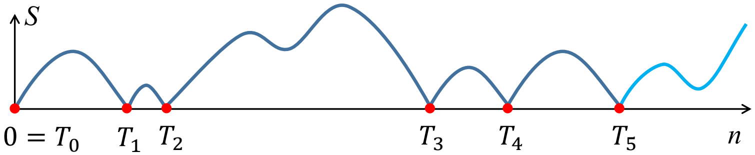
$$\begin{aligned} P_i(\sigma_{r+1} < \infty) =& P_i(\sigma_1 < \infty, \sigma_{r+1} < \infty) \\ =& \rho_i P_i(\sigma_{r+1} < \infty | \sigma_1 < \infty) = \rho_i P_i(\sigma_r < \infty) = \rho_i^{r+1}. \end{aligned}$$

- 注: 若$\rho_i < 1$, 则$\sigma_1, \sigma_2, \cdots$ 还可以取值∞. 它们不独立, 例,

$$P_i(\sigma_1 < \infty, \sigma_2 = m_2) = 0;$$

$$P_i(\sigma_1 = m_1, \sigma_2 < \infty) = P_i(\sigma_1 = m_1) \cdot \rho_i.$$

二、回访次数



- $\rho_i := P_i(\sigma < \infty) = 1$.
- $V_i = \sum_{n=0}^{\infty} \mathbf{1}_{\{X_n = i\}} = |\{n \geqslant 0 : X_n = i\}|$.
- V_i 服从(广义的)几何分布, ρ_i vs $1 - \rho_i$.

$$P(V_i \geqslant r + 1) = P_i(T_r < \infty) = \rho_i^r, \quad r \geqslant 0.$$

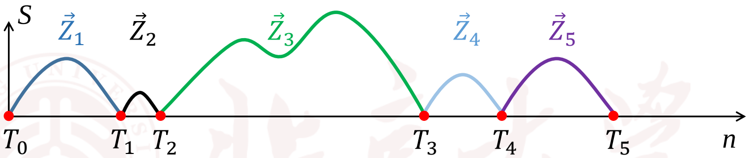
- 二择一法则, 0-1律.

  若$\rho_i = 1$, 则 $P_i(V_i = \infty) = 1$ (常返), $E_i V_i = \infty$.

  若$\rho_i < 1$, 则 $P_i(V_i < \infty) = 1$ (非常返), $E_i V_i < \infty$.

- 假设i 常返, $X_0 = i$. 令

$$\vec{Z}_r = (X_{T_{r-1}}, \cdots, X_{T_r}).$$



- 命题1.5.1. (2). $\sigma_1 = |\vec{Z}_1|, \sigma_2 = |\vec{Z}_2|, \cdots$ i.i.d..
- $\vec{Z}_1, \vec{Z}_2, \cdots$ 是离散型随机变量, 取值于$J = \bigcup_{n=1}^{\infty} J_n$,

$$J_n := \left\{ \vec{i} \in S^{n+1} : i_0 = i_n = i, \text{ 且 } i_r \neq i, \ r = 1, \cdots, n-1 \right\}.$$

其中, $\vec{i} = (i_0, i_1, \cdots, i_n)$.

命题 (命题1.5.5)

设i 常返, $X_0 = i$. 则$\vec{Z}_1, \vec{Z}_2, \cdots$ 独立同分布.

- 证: $\forall r \geqslant 2, \forall \vec{i}_1, \cdots, \vec{i}_r \in \bigcup_{n=1}^{\infty} J_n$,
  往计算$\{\vec{Z}_1 = \vec{i}_1, \vec{Z}_2 = \vec{i}_2, \cdots, \vec{Z}_r = \vec{i}_r\}$ 的概率.
- 由强马氏性,

$$P(B|C) = P(B|A, C) = P_i(\vec{Z}_1 = \vec{i}_2, \cdots, \vec{Z}_{r-1} = \vec{i}_r).$$

  其中, $A = \{\sigma = |\vec{i}_1|, X_\sigma = i\}$.
- 用归纳法, ✓.
- 游弋(excursion): $\vec{Z} = \vec{Z}_1$, 分布列如下:

$$P(\vec{Z} = \vec{i}) = p_{i_0 i_1} \cdots p_{i_{n-1} i_n}, \quad \forall n \geqslant 1, \ \vec{i} = (i_0, \cdots, i_n) \in J_n.$$

三、常返与互通类

命题 (命题1.5.6)

设i 常返且$i \to j$. 则

(a) j 常返且$j \to i$.

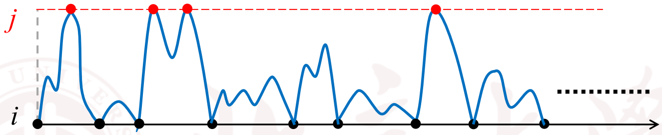
(b) 进一步, $P_i(V_j = \infty) = P_j(V_i = \infty) = 1$.

(c) 从而, $P_i(\tau_j < \infty) = P_j(\tau_i < \infty) = 1$.

- 注: $i \to j$ iff $P_i(\tau_j < \infty) > 0$, vs $P_i(\tau_j < \infty) = 1$.
- 证: 等价地, 需验证如下(i), (ii), (iii).
- (i) $P_i(V_j = \infty) = 1$. $(\Rightarrow$ (iv) $P_i(\tau_j < \infty) = 1, \Rightarrow i \to j)$.
- (ii) $P_j(V_i = \infty) = 1$. $(\Rightarrow$ (v) $P_j(\tau_i < \infty) = 1, \Rightarrow j \to i)$.
- (iii) j 常返.
- (a): (iii) & (v); (b): (i) & (ii); (c): (iv) & (v).

往证(i) $P_i(V_j=\infty)=1$.

- 设$X_0=i$, 则$\vec{Z}_1,\vec{Z}_2,\cdots$ i.i.d..



- $A_r=$ "$\vec{Z}_r$ 中出现j" .

  则$A_1,A_2,\cdots$ 相互独立, 且$P(A_r)\equiv P(A_1)\triangleq p$.

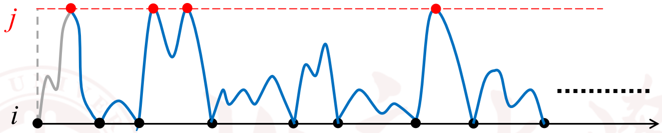
- $p>0$: 因为$\{\tau_j<\infty\}\subseteq\bigcup_r A_r$,

  所以若$p=0$, 则$P_i(\tau_j<\infty)=0$. 与$i\to j$ 矛盾!

- SLLN: $A_1,A_2,\cdots$ 发生无穷多次, $\checkmark$.

往证(ii) $P_j(V_i = \infty) = 1$.

- 由(i), $P_i(\tau_j < \infty) = 1$.



- 设$X_0 = i$. 令$Y_n = X_{\tau_j + n}$. 则$V_i = 1 + V_i^{(Y)}$. 故

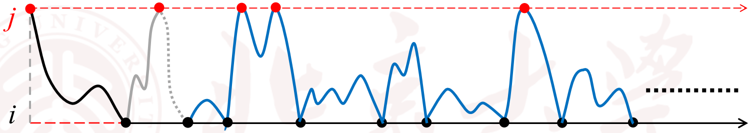
$$\{V_i = \infty\} = \{V_i^{(Y)} = \infty\}.$$

- 由强马氏性,

$$1 = P_i(V_i = \infty) = P_i(V_i^{(Y)} = \infty, \tau_j < \infty) = 1 \times P_j(V_i = \infty).$$

往证(iii) j 常返.

- 由(ii), $P_j(\tau_i < \infty) = 1$.



- 设 $X_0 = j$. 令 $Y_n = X_{\tau_i+n}$. $V_j = 1 + V_j^{(Y)}$.

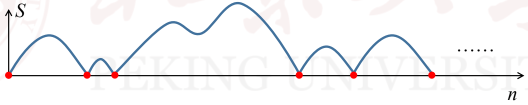
$$P_j(V_j = \infty) = P_j(\sigma_i < \infty, V_j^{(Y)} = \infty) = 1 \times P_i(V_j = \infty) = 1.$$

四、总结

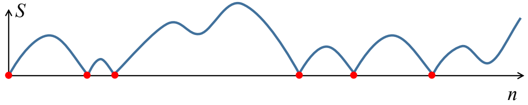
- 关键点: $\rho_i := P_i(\sigma < \infty)$ 是否等于1.
- 常返与否的判别法: 定义、击中概率、格林函数.

  常返: $P_i(V_i = \infty) = 1$, $\rho_i = 1$, $E_iV_i = \infty$.

  非常返: $P_i(V_i < \infty) = 1$, $\rho_i < 1$, $E_iV_i < \infty$.
- 常返轨道图: 由i.i.d. 游弋拼接得到.



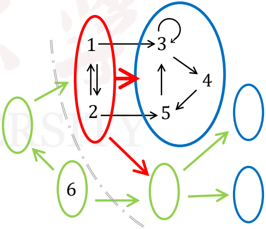
- 非常返轨道图.



- 互通类分为常返类、非常返类.

按定义判别常返与否.

- 例, 一维非对称紧邻随机游动. 由SLLN 知非常返.
- 若互通类不闭, 则非常返. 例, $\{1,2\}$.
- 若互通类是闭的, 且有限, 则常返. (§1.5 习题1 (3).)
  - 证: $P_i(\sum_{j\in S} V_j = \infty) = 1$.
    故$\exists j \in S$ 使得$P_i(V_j = \infty) > 0$.
  - $P_i(V_j = \infty) = P_i(\tau_j < \infty)P_j(V_j = \infty)$.
    故$P_j(V_j = \infty) > 0$.
  - 由0-1律, $P_j(V_j = \infty) = 1$.



- 若互通类是闭的, 但可列, 则不确定. 例, SRW.

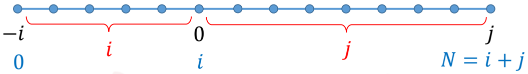
一、首步分析法与击中概率

例1.6.1. 赌徒破产问题. 求: 完胜概率.

- $\{S_n\}$: SRW.

  $\tau := \tau_0 \wedge \tau_N$.

  其中, $N = i + j$.



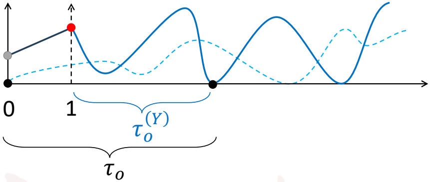
- $P(\tau_0 < \infty, \tau_N < \infty) = 1$.
- 破产: $\tau_0 < \tau_N$;　　完胜: $\tau_N < \tau_0$.
- 首步分析法: $P_k(\tau_N < \tau_0) \triangleq x_k$ 满足如下方程组:

$$x_k = \frac{1}{2} x_{k-1} + \frac{1}{2} x_{k+1}, \ 0 < k < N.$$

- 边界条件: $x_0 = 0$, $x_N = 1$.
- 解得$x_k = \frac{k}{N}$. 所求为$x_i = \frac{i}{i+j}$.

- 固定$o \in S$.
- 击中概率: $x_i = P_i(\tau_o < \infty)$.
- $\forall i \neq o$,

  $\tau_o = \tau_o^{(X)} = 1 + \tau_o^{(Y)}$.



- $\forall i \neq o$, 按X_1 的值分情况讨论:

$$x_i = P_i(\tau_o^{(Y)} < \infty) = \sum_j P_o(X_1 = j, B)$$

$$= \sum_j P_i(A) P_i(B|A) = \sum_j p_{ij} x_j.$$

- 击中概率满足如下方程组(及边界条件):

$$x_i = \sum_{j \in S} p_{ij} x_j, \quad \forall\ i \neq o; \qquad x_o = 1.$$

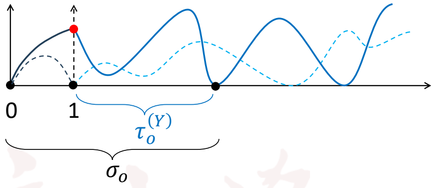
- 固定$D \subseteq S$, $x_i = P_i(\tau_D < \infty)$ 也称为击中概率.
- 击中概率满足如下方程组(及边界条件):

$$x_i = \sum_{j \in S} p_{ij} x_j, \quad \forall\ i \notin D, \quad x_i = 1, \quad \forall\ i \in D.$$

- 命题1.6.2. 击中概率是上述方程组的最小的非负解.
- 注: 若$\{\tilde{x}_i : i \in S\}$ 也是方程组的非负解, 则$\tilde{x}_i \geqslant x_i$, $\forall i$.
- 吸收态: $p_{ii} = 1$ (定义1.6.7). 吸收概率: $P_i(\tau_o < \infty)$.
  闭集D 的吸收概率: $P_i(\tau_D < \infty)$.
- 注: $\forall D \subseteq S$, 将D 改为闭集. (§1.6, 习题1).

二、击中概率与判别常返性

- 假设不可约. 固定$o \in S$.
- $\sigma_o = \sigma_o^{(X)} = 1 + \tau_o^{(Y)}$.
- $\rho_o = P_o(\sigma_o < \infty)$
  $= P_o(\tau_o^{(Y)} < \infty)$.



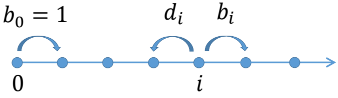
- 按X_1 的值分情况讨论:

$$\rho_o = \sum_i P_o(X_1 = i, B) = \sum_i P_o(A) P_o(B|A) = \sum_i p_{oi} P_i(\tau_o < \infty).$$

- 常返 iff $x_i = 1, \ \forall i \in S$.
  一方面, 命题1.5.6. 常返$\Rightarrow x_i = P_i(\tau_o < \infty) \equiv 1$.
  另一方面, $x_i \equiv 1 \Rightarrow \rho_o = 1$, 即常返.
- 命题1.6.4. 常返iff 方程组在$[0,1]$ 上只有恒为1 的解.

例1.6.5. 生灭链的常返性. 在本例中假设$b_i, d_i > 0,\ i \geqslant 1$.

- $x_i = P_i(\tau_0 < \infty)$ 满足的方程组:



$$x_i = b_i x_{i+1} + d_i x_{i-1},\ \forall i \geqslant 1,\ x_0 = 1.$$

- $\forall i \geqslant 1,\ b_i(x_i - x_{i+1}) = d_i(x_{i-1} - x_i)$, 故

$$x_i - x_{i+1} = (x_{i-1} - x_i)\frac{d_i}{b_i} = \cdots = (1 - x_1)\frac{d_1 \cdots d_i}{b_1 \cdots b_i}.$$

- 两边对i 求和, $1 - x_{i+1} = R_i(1 - x_1)$. 其中,

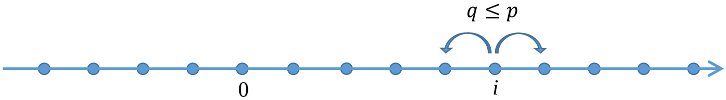
$$R_0 := 1, \quad R_i := 1 + \sum_{k=1}^{i} \frac{d_1 \cdots d_k}{b_1 \cdots b_k} \nearrow R := 1 + \sum_{k=1}^{\infty} \frac{d_1 \cdots d_k}{b_1 \cdots b_k}.$$

- 若$R < \infty$, 则取$x_1 = 1 - 1/R$, 可得不恒为1 的解. 非常返!
- 若$R = \infty$, 则$x_i = 1$. 常返!

$$1 - x_1 = \frac{1}{R_i}(1 - x_{i+1}) \leqslant \frac{1}{R_i} \xrightarrow{i\to\infty} 0.$$

例1.6.5. 生灭链的常返性(续).

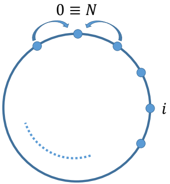
- 特例. 判断一维紧邻随机游动的常返性.



- 只看$\mathbb{Z}_-$: $b_i = q$, $d_i = p$, 常返:

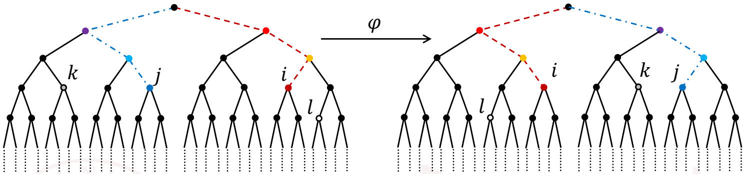
$$R := 1 + \sum_{k=1}^{\infty} \frac{d_1 \cdots d_k}{b_1 \cdots b_k} = \infty.$$

- 只看$\mathbb{Z}_+$: $b_i = p$, $d_i = q$. 若$p = q$, 则常返; 否则非常返.
- 结论: 常返iff $p = q$.
- 已得$P_0(\tau_1 < \infty) = 1$. 求: $x = P_0(\tau_{-1} < \infty)$.
  - 强马氏性$\Rightarrow P_0(\tau_{-2} < \infty) = x^2$.
  - $x = px^2 + q \Rightarrow x = \kappa$ 或1 (舍掉).



- §1.6. 习题8.: 求$y_i = P_i(\tau_N < \tau_0)$, $0 < i < N$.

例1.6.6. 规则树$\mathbb{T}^d$ 上的λ-biased 随机游动.



- i_0: i 的父结点; $i_1, \cdots, i_d$: i 的子结点.

$$p_{ij} = \frac{\lambda}{\lambda + d}, \ j = i_0; \quad p_{ij} = \frac{1}{\lambda + d}, \ j \in \{i_1, \cdots, i_d\}$$

- 球对称性: $|i| = |j|$, 则$P_i(\tau_o < \infty) = P_j(\tau_o < \infty) \triangleq a_{|i|}$.
- 方程组:

$$a_n = \frac{\lambda}{\lambda + d} a_{n-1} + \frac{d}{\lambda + d} a_{n+1}, \quad \forall n \geqslant 1; \quad a_0 = 1.$$

例1.6.6(续).

- $P_i(\tau_o < \infty) \triangleq a_{|i|}$. 方程组:

$$a_n = \frac{\lambda}{\lambda+d} a_{n-1} + \frac{d}{\lambda+d} a_{n+1}, \quad \forall n \geqslant 1; \quad a_0 = 1.$$

- 令$b_n = a_n - a_{n+1}$, $\kappa = \lambda/d$. 则

$$db_n = \lambda b_{n-1} \Rightarrow b_n = \kappa b_{n-1} = \cdots = \kappa^n b_0$$

$$\Rightarrow a_0 - a_n = b_0 + b_1 + \cdots + b_{n-1} = \left(1 + \kappa + \cdots + \kappa^{n-1}\right) b_0.$$

- 设$\lambda < d$. 取$b_0 = 1 - \kappa$, 的最小的非负解$a_n = \kappa^n$. 非常返!
- 设$\lambda \geqslant d$. 为保证$a_n \geqslant 0$, $\forall n$, 只能取$b_0 = 0$. 故最小的非负解为$a_n \equiv 1$. 常返!

§1.7 格林函数

一、格林函数与判别常返性

- 回顾: i 常返iff $E_iV_i = \infty$. 其中,

$$V_i = \sum_{n=1}^{\infty} \mathbf{1}_{\{X_n=i\}}.$$

- 格林函数:

$$G_{ij} := E_i \sum_{n=0}^{\infty} \mathbf{1}_{\{X_n=j\}} = \sum_{n=0}^{\infty} P_i(X_n = j) = \sum_{n=0}^{\infty} p_{ij}^{(n)},$$

例1.7.1. d 维简单随机游动$\{S_n\}$ 的常返性.

- $d=1$, 常返.
  - $p_{00}^{(2n+1)}=0$,

$$p_{00}^{(2n)}=C_{2n}^{n}\frac{1}{2^{2n}}=\frac{(2n)!}{n!n!}\cdot\frac{1}{2^{2n}}\approx\frac{(\frac{2n}{e})^{2n}\sqrt{2\pi 2n}}{(\frac{n}{e})^{2n}2\pi n}\cdot\frac{1}{2^{2n}}=\frac{1}{\sqrt{\pi n}}.$$

- $d=2$, 常返.
  - $p_{00}^{(2n+1)}=0$,

$$p_{00}^{(2n)}=\sum_{n_1+n_2=n}\frac{(2n)!}{(n_1!)^2(n_2!)^2}\frac{1}{4^{2n}}$$

$$=C_{2n}^{n}\frac{1}{2^{2n}}\left(\sum_{m=0}^{n}C_n^mC_n^{n-m}\right)\frac{1}{2^{2n}}=\left(C_{2n}^{n}\frac{1}{2^{2n}}\right)^2\approx\frac{1}{\pi n}.$$

$d=3$, 非常返.

- $p_{00}^{(2n)} = \sum_{n_1+n_2+n_3=n} \frac{(2n)!}{(n_1!)^2(n_2!)^2(n_3!)^2} \frac{1}{6^{2n}}$.
- 先确定方向:

$$P(K=2n_1, L=2n_2, M=2n_3) = \frac{(2n)!}{(2n_1)!(2n_2)!(2n_3)!} \frac{1}{3^{2n}}.$$

- 在确定前后:

$$P(B_1) = C_{2n_1}^{n_1} \frac{1}{2^{2n_1}}, P(B_2) = C_{2n_2}^{n_2} \frac{1}{2^{2n_2}}, P(B_3) = C_{2n_3}^{n_3} \frac{1}{2^{2n_3}}.$$

- $p_{00}^{(2n)} = \sum_{n_1+n_2+n_3=n} P(A)P(B_1)P(B_2)P(B_3)$.
- 直观: SLLN, $n_1, n_2, n_3 \approx n/3 \Rightarrow p_{00}^{(2n)} \approx \frac{1}{\sqrt{\pi n/3}^3}$.
- 具体证明: 令

$$\vec{N} = \{\vec{n} = (n_1, n_2, n_3) \in \mathbb{Z}_+^3 : n_1 + n_2 + n_3 = n\},$$

$$I = \{\vec{n} \in \vec{N} : n_i \geqslant n/6, \ \forall i\}.$$

$d=3$, 非常返(续).

- 若$\vec{n}\in I$ 中, 则$P(B_i)\leqslant \tilde{C}/\sqrt{\pi n_i}\leqslant C^{1/3}/\sqrt{n}$. 故

$$\sum_{\vec{n}\in I} P(A)P(B_1)P(B_2)P(B_3)\leqslant \sum_{\vec{n}\in I} P(A)\frac{C}{n^{3/2}}\leqslant \frac{C}{n^{3/2}}.$$

- 回顾: $A=\{K=2n_1, L=2n_2, M=2n_3\}$.

  注: K, L, M 都服从$B(2n,1/3)$, 但它们不独立.

- 令$J=\vec{N}\backslash I=\{\vec{n}\in\vec{N}:\exists i \text{ s.t. } n_i<n/6\}$. (§1.7. 习题9 (1))

$$\sum_{\vec{n}\in J} P(A)P(B_1)P(B_2)P(B_3)\leqslant \sum_{\vec{n}\in J} P(A)$$

$$=3P(K<\frac{n}{3})\leqslant 3P(|K-\frac{n}{3}|>\frac{n}{3})\leqslant 3\frac{E(K-n/3)^4}{(n/3)^4}\leqslant \frac{\hat{C}}{n^2}.$$

- $p_{00}^{(2n)}\leqslant \star\star+\star\star\leqslant \frac{C+1}{n^{3/2}},\quad n\geqslant n_0.$

$d \geqslant 3$, 非常返.

- $p_{00}^{(2n)} = \sum_{n_1+\cdots+n_d=n} \frac{(2n)!}{(n_1!)^2\cdots(n_d!)^2} \frac{1}{2d^{2n}}$.
- $P(K_1 = 2n_1, \cdots, K_d = 2n_d) = \frac{(2n)!}{(2n_1)!\cdots(2n_d)!} \frac{1}{3^{2n}}$.
- $P(B_i) = C_{2n_i}^{n_i} \frac{1}{2^{2n_i}}$.
- $p_{00}^{(2n)} = \sum_{n_1+\cdots+n_d=n} \underbrace{P(A)P(B_1)\cdots P(B_d)}$.
- 类似地, 取$\vec{N}$; $I = \{\vec{n} \in \vec{N} : n_i \geqslant n/(2d), \forall i\}$; J.
- $\sum_{\vec{n}\in I} \star\star \leqslant \sum_{\vec{n}\in I} P(A) \left(\tilde{C}/\sqrt{\pi \cdot n/(2d)}\right)^d \leqslant \tilde{C}_d/n^{d/2}$.
- $\sum_{\vec{n}\in J} \star\star \leqslant \sum_{\vec{n}\in J} P(A) \leqslant d \cdot P(K_1 < n/d) \leqslant \hat{C}_d/n^2$.
- $p_{00}^{(2n)} \leqslant \sum_{\vec{n}\in I} \star\star + \sum_{\vec{n}\in J} \star\star \leqslant C_d \frac{1}{n^2}, \quad n \geqslant n_0$.
- §1.7 习题9 (2): $\forall a > 0$, $K_1 = \xi_1 + \cdots + \xi_{2n} \sim B(2n, 1/d)$,

$$P(K_1 < n/d) \leqslant Ee^{a(n/d-K_1)} \leqslant \varphi(a)^n. \quad 取\ a\ \text{s.t.}\ \varphi(a) < 1.$$

二、格林函数的其他应用

- 推论1.7.3. 若i 常返且$i \to j$, 则j 常返.　(注: 命题1.5.6.)
- 命题1.7.4. $G_{ij} = P_i(\tau_j < \infty)G_{jj}$.
- 证: 在$\tau_j < \infty$ 上, $V_j = V_j^{(Y)}$.
- 推论1.7.5. 若j 非常返, 则$\lim_{n\to\infty} p_{ij}^{(n)} = 0$, $\forall i$.
- 证: $\sum_n p_{ij}^{(n)} = G_{ij} \leqslant G_{jj} < \infty$.
- 推论1.7.6. 若π 是不变分布, j 非常返, 则$\pi_j = 0$.
- 证: $(\sum_n \pi \mathbf{P}^n)_j = \infty \cdot \pi_j$,

  $(\pi \sum_n \mathbf{P}^n)_j = \sum_i \pi_i G_{ij} \leqslant G_{jj} < \infty$.

三、有限区域中的格林函数

- 记$\tau = \tau_{D^c}$,

$$G_{ij}^{(D)} := E_i \sum_{n=0}^{\tau-1} \mathbf{1}_{\{X_n=j\}}, \quad i,j \in D.$$

- 固定$o \in D$, $x_i = G_{io}^{(D)}$ 满足的方程组(及边界条件):

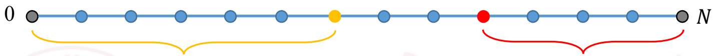
$$x_i = \sum_{j \in S} p_{ij} x_j, \ \forall i \in D \backslash \{o\}; \ \ x_o = 1 + \sum_{j \in S} p_{oj} x_j; \quad x_i = 0, \ i \in D^c.$$

- 注:

$$G_{ij}^{(D)} := \sum_{n=0}^{\tau-1} P_i(X_n = j).$$

例1.7.7. 一维SRW. $N \geqslant 3$, $D = \{1, \cdots, N-1\}$. 则

$$G_{ij}^{(D)} = \begin{cases} \frac{2}{N} \cdot i(N-j), & 0 < i \leqslant j < N, \\ \frac{2}{N} \cdot j(N-i), & 0 < j < i < N, \end{cases}$$

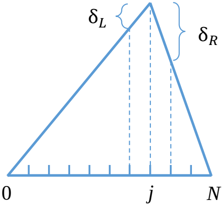


- 固定j. $x_i = G_{ij}^{(D)}$ 满足如下方程组(及边界条件):

$$\begin{cases} x_i = \frac{1}{2}(x_{i-1} + x_{i+1}), & 0 < i < N, i \neq j; \\ x_j = 1 + \frac{1}{2}(x_{j-1} + x_{j+1}), & x_0 = x_N = 0. \end{cases}$$

- $\delta_L = \frac{1}{j} x_j$; $\delta_R = \frac{1}{N-j} x_j$.
- $0 = -1 + \delta_L/2 + \delta_R/2 \Rightarrow x_j = \frac{2j(N-j)}{N}$.
- $x_i = \frac{i}{j} x_j = \frac{2i(N-j)}{N}, \quad 0 < i < j;$

  $x_i = \frac{N-i}{N-j} x_j = \frac{2j(N-i)}{N}, \quad j < i < N.$



例1.7.8. 一维SRW. 则$E_i(\tau_0 \wedge \tau_N) = i(N-i),\ \ 0 < i < N$.

- 方法一、$\tau = \tau_{D^c}$. $y_i = E_i\tau$ 满足如下方程组(及边界条件):

$$y_i = 1 + \sum_{j\in S} p_{ij} y_j,\ \forall\ i \in D; \quad y_i = 0,\ \forall i \notin D.$$

- $y_i = 1 + \frac{1}{2}y_{i+1} + \frac{1}{2}y_{i-1},\ 1 \leqslant i \leqslant N-1.$
- 令$\delta_i = y_i - y_{i-1}$, 则$\delta_i = 2 + \delta_{i+1}$.
- $\delta_i = \delta_{i-1} - 2 = \cdots = \delta_1 - 2(i-1)$.
- $\delta_1 = N-1$:

$$\sum_{i=1}^{N} \delta_i = \tau_N - \tau_0 = 0 \Rightarrow N\delta_1 - N(N-1) = 0.$$

- $\delta_i = N - 1 - 2(i-1)$, 故

$$y_i = y_0 + \delta_1 + \cdots + \delta_i = i(N-1) - i(i-1) = i(N-i).$$

- 注: 令$N \to \infty$, $E_i\tau_0 = \lim_{N\to\infty} E_i(\tau_0 \wedge \tau_N) = \infty$.

例1.7.8(续). 一维SRW. 则$E_i(\tau_0 \wedge \tau_N) = i(N-i),\ \ 0 < i < N.$

- 方法二、先得到$G_{ij}^{(D)}$, $i, j \in D$.
- 取$X_0 = i \in D$.

$$\tau = \sum_{n=0}^{\tau-1} 1 = \sum_{n=0}^{\tau-1} \sum_{j\in D} \mathbf{1}_{\{X_n=j\}} = \sum_{j\in D} \sum_{n=0}^{\tau-1} \mathbf{1}_{\{X_n=j\}},$$

$$\Rightarrow E_i\tau = \sum_{j\in D} E_i \sum_{n=0}^{\tau-1} \mathbf{1}_{\{X_n=j\}} = \sum_{j\in D} G_{ij}^{(D)}.$$

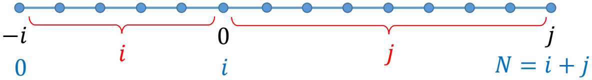
- 特别地, 一维SRW. 则

$$E_i(\tau_0 \wedge \tau_N) = \sum_{j=1}^{i} \frac{2}{N} j(N-i) + \sum_{j=i+1}^{N-1} \frac{2}{N} i(N-j)$$

$$= \frac{2}{N} \frac{i(i+1)(N-i)}{2} + \frac{2}{N} \frac{i(N-i-1)(N-i)}{2} = i(N-i).$$

四、Wald 等式

- 一维SRW $\{S_n\}$. $X_0 = 0$, $\tau = \tau_j \wedge \tau_{-i}$.



- S_τ 的分布:

$$P_0(S_\tau = j) = P_i(\tau_N < \tau_0) = \frac{i}{N},$$

$$P_0(S_\tau = -i) = P_i(\tau_0 < \tau_N) = \frac{N-i}{N}.$$

- $E_0 S_\tau = 0$.
- $E_0 S_\tau^2 = E_0 \tau$.

$$ES_\tau^2 = j^2 \cdot \frac{i}{i+j} + i^2 \cdot \frac{j}{i+j} = i \cdot j,$$

$$E_0 \tau = E_i(\tau_0 \wedge \tau_N) = i \cdot j.$$

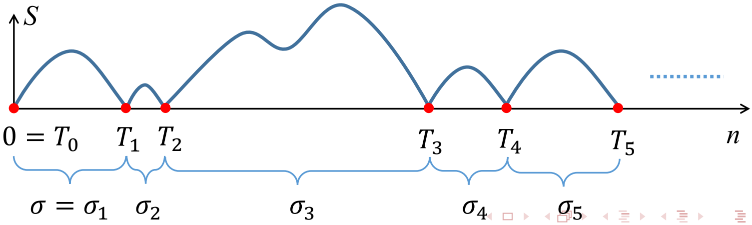
- τ 为首达时, 首中时, 第r 次访问的时刻, $\cdots$
- 定理1.7.9 (Wald 等式).
  若$E\tau$ 与$E\xi$ 存在(且为实数), 则$ES_\tau = E\tau \cdot E\xi$.
- 定理1.7.13 (Wald 第二等式).
  设$E\tau$ 与$E\xi^2$ 存在且$E\xi = 0$. 若存在$M \geqslant 0$ 使得$|S_n| \leqslant M$, $\forall n \leqslant \tau$. 则$ES_\tau^2 = E\tau \cdot E\xi^2$.

一、频率的极限

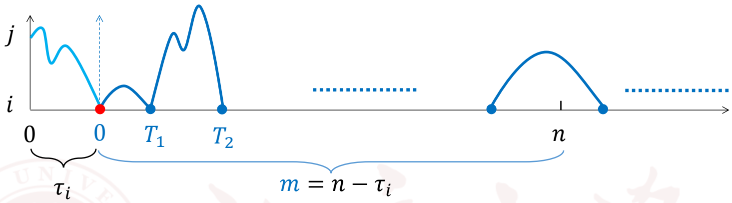
定理 (定理1.8.1)

假设**P** 不可约. 则 $P\left(\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}\mathbf{1}_{\{X_m=i\}}=\frac{1}{E_i\sigma_i}\right)=1.$

- 注: $V_i(n)/n$ 为访问i 的频率.
- 证明: 若非常返. 则左右都为0. 下设常返.
- 假设$X_0=i$. 用更新定理即可.



- 假设$X_0 = j$. 常返, 故$P_j(\tau_i < \infty) = 1$.



- $\tau = \tau_i$, $\tilde{\omega} = (Z_0(\omega), \cdots, Z_\tau(\omega))$, $\hat{\omega} = (Y_0(\omega), Y_1(\omega), \cdots)$.
  $\omega = (\tilde{\omega}, \hat{\omega})$. $\quad n = m + \tau_i(\tilde{\omega})$, $\quad V_i(n, \omega) = V_i(m, \hat{\omega})$.
- $n \to \infty$ iff $m \to \infty$:

$$\frac{1}{n} V_i(n, \omega) = \frac{m}{n} \times \frac{1}{m} V_i(m, \hat{\omega}) \to \frac{1}{E_i \sigma_i}.$$

- 假设$X_0 \sim \mu$:

$$P\left(\lim_{n\to\infty} \frac{1}{n} V_i(n) = \frac{1}{E_i \sigma_i}\right) = \sum_j \mu_j P_j(A) = 1.$$

命题 (命题1.8.2, 频率→ 概率)

假设**P** 不可约, π 为不变分布. 则

$$\pi_i = \frac{1}{E_i\sigma_i} > 0, \quad \forall i \in S.$$

- $\pi_i > 0$: $\pi_i = \sum_j \pi_j p_{ji}^{(n)} \geqslant \pi_o p_{oi}^{(n)} > 0$.
- 由定理1.8.1, $\frac{1}{n}V_i(n) \to \frac{1}{E_i\sigma_i}$, a.s..
- 再由有界收敛定理,

$$\lim_{n\to\infty} E_\pi \frac{1}{n}V_i(n) = \frac{1}{E_i\sigma_i}.$$

- π 是不变分布. 故

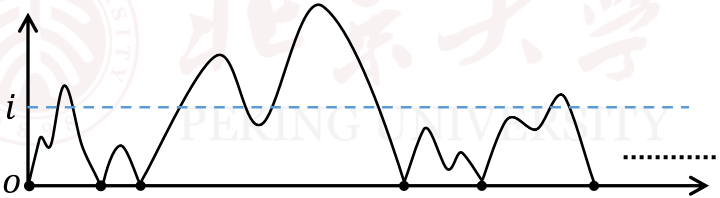
$$E_\pi \frac{1}{n}V_i(n) = E_\pi \frac{1}{n}\sum_{m=0}^{n-1} \mathbf{1}_{\{X_m=i\}} = \frac{1}{n}\sum_{m=0}^{n-1} P_\pi(X_m = i) = \pi_i.$$

- 注: 设不可约. 不变分布若存在, 则唯一(表达式).

二、正常返与不变分布

- 定理1.8.1 设不可约. 则访问频率$\frac{1}{n}V_i(n) \to \frac{1}{E_i\sigma_i}$, a.s..
- 正常返: $E_i\sigma_i < \infty$. 访问频率为正. (定义1.8.3)

  零常返: $P_i(\sigma_i < \infty) = 0$ 但$E_i\sigma_i = \infty$. 访问频率为0.
- 再看频率: 固定o. 记$\sigma = \sigma_o$.



$$i\text{出现的频率} \approx \frac{V_i^{(1)} + \cdots + V_i^{(r)}}{\sigma_1 + \cdots + \sigma_r} \approx \frac{E_o V_i^{(1)}}{E_o\sigma} \propto E_o V_i^{(1)}.$$

命题 (命题1.8.4)

假设o 常返. 令$\mu_i = E_o \sum_{n=0}^{\sigma_o - 1} \mathbf{1}_{\{X_n = i\}}$. 则$\mu$ 是满足不变方程:

$$\sum_j \mu_j p_{ji} = \mu_i \quad \forall i \in S.$$

- 记$\sigma = \sigma_o$. 注: $\mu_i \neq \sum_{n=0}^{\sigma-1} P_o(X_n = i)$.
- $\mu_o = 1$. 当$j \neq o$ 时, $\sum_{n=0}^{\sigma-1} \mathbf{1}_{\{X_n = i\}} = \sum_{n=0}^{\infty} \mathbf{1}_{\{n<\sigma, X_n = j\}}$, 故

$$\mu_j = E_o \sum_{n=0}^{\infty} 1_{\{n<\sigma, X_n = j\}} = \sum_{n=0}^{\infty} P_o(n < \sigma, X_n = j).$$

- $\{n < \sigma\} = \{\sigma > n\} = \{X_1, \cdots, X_{n-1} \neq o\} \cap \{X_n \neq o\}$.
- $P_o(n < \sigma, X_n = j) = P_o(CA)$.
  其中, $A = \{X_n = j\}$, $C = \{X_1, \cdots, X_{n-1} \neq o\}$.

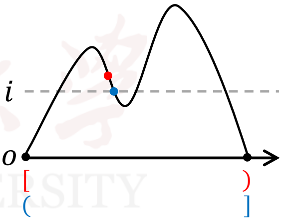
- 目标: 证明LHS $=\sum_j \mu_j p_{ji} = \mu_i \quad \forall i\in S$.
- 已证: $\mu_j = \sum_{n=0}^{\infty} P_o(\underline{n<\sigma, X_n = j})$,

  其中, $\underline{\star\star} = AC$, $A=\{X_n=j\}$, $C=\{X_1,\cdots,X_{n-1}\neq o\}$.
- $p_{ji} = P_o(X_{n+1}=i|AC)$. 于是,

$$\text{LHS} = \sum_j \sum_{n=0}^{\infty} P_o(AC)P_o(X_{n+1}=j|AC)$$

$$=P_o(n<\sigma, X_n=j, X_{n+1}=i)$$

$$=\sum_{n=0}^{\infty} P_o(n<\sigma, X_{n+1}=i).$$



- LHS $=\sum_{n=0}^{\infty} E_o \mathbf{1}_{\{n<\sigma, X_{n+1}=i\}} = E_o \sum_{n=0}^{\sigma-1} \mathbf{1}_{\{X_{n+1}=i\}}$.
- LHS $= \underset{\sim\sim\sim}{E_o\sum_{m=1}^{\sigma} \mathbf{1}_{\{X_m=i\}}} = \mu_i$, 因为$\underset{\sim}{\star\star} = \sum_{n=0}^{\sigma-1} \mathbf{1}_{\{X_n=i\}}$.
- 注: $\sigma=\infty$, $i=o$ 时, $=$ 不成立.
- 注: $\mu_j<\infty$, 互通则$\mu_j>0$. (因为$\mu_k \geqslant \mu_i p_{ik}^{(n)}$.)

定理 (定理1.8.5)

假设不可约. 则下面三条等价:

(1) 所有状态正常返, (2) 存在正常返态, (3) 存在不变分布.

特别地, 不变分布的表达式为$\pi_i = \frac{1}{E_i\sigma_i}$, $\forall i \in S$. (✓)

- (1) $\overset{\checkmark}{\Rightarrow}$ (2).
- (2) $\Rightarrow$ (3): 任取正常返态o. $\mu_i = E_o \sum_{n=0}^{\sigma_o - 1} \mathbf{1}_{\{X_n = i\}}$ 为不变测度, 可归一化. ✓.
- (3) $\Rightarrow$ (1): (i) 由命题1.8.2, ★★ 成立; (ii) 由不变方程及不可约, $\pi_i > 0$, $\forall i$. 故(1) 成立. □
- 注: 互通类分为正常返类、零常返类、非常返类.

推论 (推论1.8.7)

若S 有限且不可约, 则不变分布存在.

- 证: 由S 不可约, 对任意初始分布μ

$$\frac{1}{n}\sum_{m=0}^{n-1}\mathbf{1}_{\{X_m=i\}}\to\frac{1}{E_i\sigma_i},\quad \text{a.s..}$$

- S 有限, 故

$$1=\sum_i\frac{1}{n}\sum_{m=0}^{n-1}\mathbf{1}_{\{X_m=i\}}\to\sum_i\frac{1}{E_i\sigma_i}.$$

- 存在i 使得$E_i\sigma_i<\infty$, 即正常返. 故✓.
- 注: 第1.2 节中提到用代数或分析的方法求不变分布.

补充知识.

- 引理1.8.19., 设$\mathbf{P}$ 不可约、常返. 取定o. 则

$$\mu_i = E_o \sum_{n=0}^{\sigma_o-1} \mathbf{1}_{\{X_n=i\}}$$

是满足不变方程及边界条件$\mu_o = 1$的唯一的解.

- 注: 若非常返, 则不变测度可能不唯一(注1.6.5)
- 推论1.8.20. $\exists$ 不能归一化的不变测度λ, 则不变分布不存在.
- 证: 若非常返, 则不变分布不存在, 否则$\pi_i \equiv 0$ (推论1.7.6). 若常返, $\pi_i \propto \mu_i$, $\lambda_i \propto \mu_i$. 但λ 不能归一化, 矛盾!
- 例. $\mathbb{Z}^d$ 上的随机游动没有不变分布.
- 注: S 划分为正常返类$\{C_\alpha : \alpha \in I\}$、零常返类、非常返类. 则所有不变分布形如$\sum_\alpha p_\alpha \pi^{(\alpha)}$, 其中$\{p_\alpha : \alpha \in I\}$ 为分布列.

三、遍历定理

- 假设不可约、正常返. 频率→ (不变分布对应的)概率:

$$\frac{1}{n}\sum_{m=0}^{n-1}\mathbf{1}_{\{X_m=i\}}\xrightarrow{\text{a.s.}}\pi_i=\frac{1}{E_i\sigma_i}=\frac{E_oV_i^{(1)}}{E_o\sigma_o},\quad \forall i.$$

- 再由有界收敛定理:

$$\frac{1}{n}\sum_{m=0}^{n-1}P(X_m=i)\to\pi_i,\quad \forall i.$$

- 引理1.8.8: 设都是分布列; $\lim_{n\to\infty}\pi_i^{(n)}=\pi_i$, $\forall i\in S$. 则

$$\lim_{n\to\infty}\sum_{i\in S}|\pi_i^{(n)}-\pi_i|=0.$$

- 命题1.7.9. $\sum_i|\pi_i^{(n)}-\pi_i|\to 0$,

$$\sum_{i\in S}\left|\frac{1}{n}\sum_{m=0}^{n-1}p_{ji}^{(m)}-\pi_i\right|\to 0,\quad \sum_{i\in S}\left|\frac{V_i(n)}{n}-\pi_i\right|\xrightarrow{\text{a.s.}}0.$$

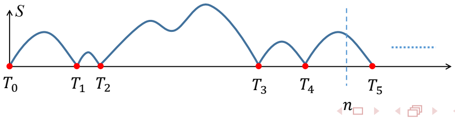
定理 (遍历定理, 定理1.8.10)

假设不可约、正常返. 若$\sum_i \pi_i |f(i)| < \infty$, 则

$$\frac{1}{n}\sum_{m=0}^{n-1} f(X_m) \xrightarrow{\text{a.s.}} \sum_i \pi_i f(i).$$

- 假设f 有界, $M := \sup_i |f(i)| < \infty$.
- LHS $= \frac{1}{n}\sum_i V_i(n) f(i) = \sum_i \frac{V_i(n)}{n} \cdot f(i)$.
- $|\text{LHS} - \text{RHS}| \leqslant M \sum_i \left|\frac{V_i(n)}{n} - \pi_i\right| \xrightarrow{\text{a.s.}} 0$.
- 补充知识. 一般情况, 固定o, 设$X_0 = o$.

$$\xi_r = \sum_{n=T_{r-1}}^{T_r - 1} f(X_n), \quad \text{LHS} \approx \frac{\xi_1 + \cdots + \xi_r}{r} \cdot \frac{r}{n}.$$



四、应用

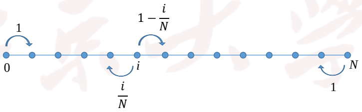
1. 表达式$\pi_i = \frac{1}{E_i\sigma_i}$ 的应用

例1.8.11. Ehrenfest模型. N 个球, 两个纸箱A, B.

每次独立地随机选一个球, 把它换到另一个纸箱中.

- $X_n = n$ 次操作后纸箱A 中的球的个数. 生灭过程,

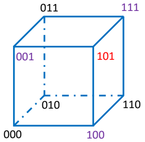
$$\pi_i = C_N^i \frac{1}{2^N}.$$



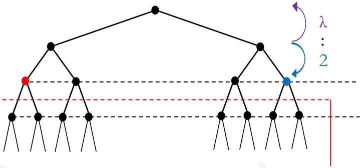
- 设$N = 2M$, 则

$$E_0\sigma_0 = \frac{1}{\pi_0} = 2^{2M}, \quad E_M\sigma_M = \frac{1}{\pi_M} = \frac{2^{2M}}{C_{2M}^M} \approx \sqrt{\pi M}.$$

- $\tilde{S} = \{0,1\}^N$. 图$\tilde{S}$ 上随机游动.
- 思考: 第r 个球的运动.



例1.8.14. $\mathbb{T}^d$ 上λ-biased 随机游动:



- 球对称性:

  若$|i| = |j|$, 则

  $E_i\sigma_i = E_j\sigma_j \Rightarrow \pi_i = c_{|i|}$.

- 取$A = \{i : |i| \geqslant n\}$.

$$(d^{n-1}c_{n-1}) \cdot \frac{d}{\lambda + d} = (d^n c_n) \cdot \frac{\lambda}{\lambda + d} \Rightarrow c_{n-1} = \lambda c_n.$$

- $c_n = \frac{1}{\lambda} \cdot c_{n-1} = \cdots = \frac{1}{\lambda^n} \cdot c_0$.
- 归一化:

$$\sum_{n=0}^{\infty} d^n c_n = 1 \Rightarrow c_0 = \frac{1}{R}, \quad 其中\ R = \sum_{n=0}^{\infty} \left(\frac{d}{\lambda}\right)^n.$$

- $\lambda > d \Leftrightarrow$ iff 正常返. 又知$\lambda \geqslant d \Leftrightarrow$ 常返, (例1.6.6.). 故

  $\lambda > d$: 正常返; $\lambda = d$: 零常返; $\lambda < d$: 非常返.

2. 遍历定理的应用

例1.8.15. 假设不可约、正常返. 则

$$\frac{1}{n}\sum_{m=0}^{n-1} 1_{\{X_m=i, X_{m+1}=j\}} \xrightarrow{\text{a.s.}} \pi_i p_{ij}.$$

- 方法一、$\{Y_n = (X_n, X_{n+1})\}$ 是马氏链, $\tilde{S} = \{(i,j) : p_{ij} > 0\}$; 转移概率: $p_{(i,j)(j,k)} = p_{jk}$, 不变分布: $\mu_{(i,j)} = \pi_i p_{ij}$. 故,

$$\star\star = \frac{1}{n}\sum_{m=0}^{n-1} 1_{\{Y_m=(i,j)\}} \xrightarrow{\text{a.s.}} \mu_{(i,j)} = \pi_i p_{ij}.$$

- 方法二: 假设n 之前完成r 次从i 出发并回到i 的游弋,

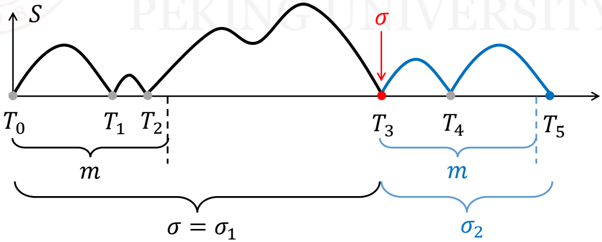
$$\frac{r}{n} \times \frac{1}{r}\sum_{s=0}^{r-1} 1_{\{X_{T_s+1}=j\}} \xrightarrow{\text{a.s.}} \pi_i p_{ij}.$$

例1.8.16. 设不可约、正常返. 令$\sigma := \inf\{n \geqslant m : X_n = i\}$. 证明:

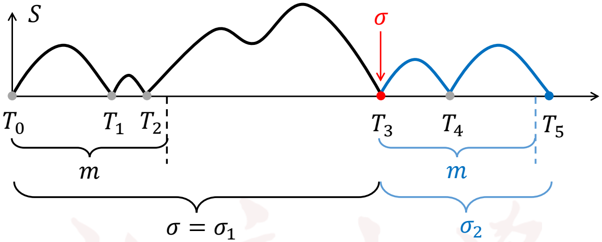
$$E_i \sum_{n=0}^{\sigma-1} 1_{\{X_n=j\}} = \pi_j E_i \sigma.$$

- 方法一、仿照命题1.7.5., 证明★ 给出不变测度.
- 方法二、$X_0 = i$. 解读频率:

$$i\text{出现的频率} \approx \frac{V_j^{(1)} + \cdots + V_j^{(r)}}{\sigma_1 + \cdots + \sigma_r} \approx \frac{E_i V_j^{(1)}}{E_i \sigma} = \pi_j.$$



例1.8.16(续).



- 令$Y_n = X_{\sigma+n}$. 则$\{Y_n\}$ 是从i 出发的马氏链, 且与$\vec{Z} = (X_0, X_1, \cdots, X_\sigma)$ 独立.
- 将σ 换成T_r 得到$\{Y^{(r)}\}$ 与$\vec{Z}^{(r)}$.
- $D_r := \{\sigma = T_r\} = \Big\{|\{n < m : X_n = i\}| = r\Big\}, r = 1, \cdots, m.$
- $B = \{\vec{Y} = \vec{j}\}$ (注: 有限维), $C = \{\vec{C} = \vec{i}\}$; 类似定义$B_r$, C_r.

$$P_{D_r}(BC) = P_{D_r}(B_rC_r) = P(\vec{X} = \vec{j})P(C_r) = P(\vec{X} = \vec{j})P_{D_r}(C).$$

五、总结

- 正常返类iff 不变分布存在. 表达式.

$$E_i\sigma_i < \infty, \quad \pi_i = \frac{1}{E_i\sigma_i}$$

- 频率$\to$ 概率, 遍历定理(时间平均= 空间平均).

$$\frac{1}{n}\sum_{m=0}^{n-1} \mathbf{1}_{\{X_m=i\}} \xrightarrow{\text{a.s.}} \pi_i, \qquad \frac{1}{n}\sum_{m=0}^{n-1} f(X_m) \xrightarrow{\text{a.s.}} \sum_i \pi_i f(i).$$

- $\mu_i = E_o \sum_{n=0}^{\sigma_o-1} \mathbf{1}_{\{X_n=i\}}$, $i \in S$ 满足不变方程, (证明方法).
- 应用1: 通过π_i 求$\frac{1}{E_i\sigma_i}$. 或者, 反过来.
- 应用2. 用轨道的i.i.d. 结构解读频率/概率.

一、正常返、非周期情形

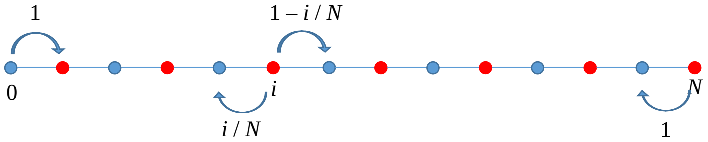
- 假设：不可约、正常返. 已知

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=0}^{n-1}P(X_m=i)=\pi_i$$

- 问：$P(X_n=i)\to\pi_i$?
- 反例. 例1.8.11. Ehrenfest模型.

$S=\{0,1,\cdots,N\}$. $\pi_i=C_N^i 2^{-N}$, $\forall i$.

取$X_0=0$. $P(X_{2n}=1)\equiv 0$, $\forall n$.



- 必要条件：n 充分大时，$p_{ij}^{(n)}>0$.

- 定义1.9.1. 若$\exists N$ 使得$p_{ii}^{(n)} > 0, \forall n \geqslant N$, 则称$i$ 为非周期的.
- 注: 若$p_{ii} > 0$, 则非周期.
- 推论1.9.2. 设不可约, 存在非周期态, 则所有状态非周期, 且$\forall i, j, \exists N$ 使得

$$p_{ij}^{(n)} > 0, \quad \forall n \geqslant N.$$

- 证: 设i 非周期.
- j 非周期, 取$N_j = N_i + r + s$ 即可:
  存在$r = r_{ij}, s = s_{ij}$ 使得$p_{ij}^{(r)}, p_{ji}^{(s)} > 0$, 于是

$$p_{jj}^{(r+s+n)} \geqslant p_{ji}^{(s)} p_{ii}^{(n)} p_{ij}^{(r)} > 0, \quad \forall n \geqslant N_i.$$

- 取$N_{ij} = N_i + r$ 即可:

$$p_{ij}^{(n)} \geqslant p_{ii}^{(m)} p_{ij}^{(r)} > 0, \quad \forall n \geqslant N_i + r$$

- 注: 若$i \leftrightarrow j$, 则i 非周期$\Rightarrow j$ 非周期. 非周期类.

定理 (定理1.9.3. 强遍历定理)

假设不可约、正常返、非周期, 不变分布为π. 则$\forall\mu$,

$$\lim_{n\to\infty}\sum_j |P_\mu(X_n=j)-\pi_j|=0. \quad \text{特别地, } \lim_{n\to\infty} p_{ij}^{(n)}=\pi_j.$$

- $\{Z_n=(W_n,Y_n)\}$: 状态空间与转移概率如下

$$\tilde{S}=S\times S, \quad r_{(i,j)(k,\ell)}=p_{ik}p_{j\ell}.$$

- 初分布记为ν, 暂定.
- Step 1 ~ 3. $\{Z_n\}$ 不可约、非周期、正常返($\tilde{\pi}$ 为不变分布):

$$r_{(i,j)(k,\ell)}^{(n)}=p_{ik}^{(n)}p_{j\ell}^{(n)}>0, \quad \forall n\geqslant N_{ik}\vee N_{j\ell};$$

$$\tilde{\pi}=\pi\times\pi, \quad \text{i.e. } \tilde{\pi}_{i,j}=\pi_i\pi_j, \ \forall (i,j)\in\tilde{S}.$$

- 且$\{W_n\}$, $\{Y_n\}$ 都是以$\mathbf{P}$ 为转移矩阵的马氏链.

- Step 4. $\{Z_n\}$ 常返, 故$P(\tau<\infty)=1$, 其中

$$\tau := \inf\{n \geqslant 0 : W_n = Y_n\} = \tau_D, \quad D := \{(i,i) : i \in S\}.$$

- Step 5. $P(W_n=j,\tau\leqslant n)=P(Y_n=j,\tau\leqslant n)$:
按相遇的时间、地点分情况讨论. $\forall 0\leqslant m\leqslant n$, $i\in S$,

$$\begin{aligned} & P(W_n = j, \tau = m, Z_m = (i,i)) \\ =& P(*,*)\underset{\sim\sim\sim}{P_{(i,i)}(W_{n-m}=j)} = P(*,*)\underset{\sim\sim\sim}{P_{(i,i)}(Y_{n-m}=j)} \\ =& P(Y_n = j, \tau = m, Z_m = (i,i)). \end{aligned}$$

- 注: $\underset{\sim\sim}{\star\star} = \sum_k p_{ij}^{(n-m)} p_{ik}^{(n-m)} = p_{ij}^{(n-m)}$.

- Step 6. $\sum_j |P(W_n = j) - P(Y_n = j)| \leqslant 2P(\tau > n)$:

$$P(W_n = j) = P(W_n = j, \tau \leqslant n) + P(W_n = j, \tau > n)$$

$$\text{故 } P(W_n = j) - P(Y_n = j) = \star - P(Y_n = j, \tau > n).$$

- 初分布ν 取为$\mu \times \pi$, 即$\nu_{(i,j)} = \mu_i \pi_j$, $\forall i, j$.
则$\{W_n\}$ 和$\{Y_n\}$ 的初分布分别为μ, π. 故

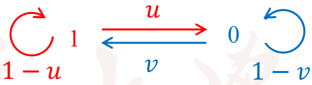
$$P_\nu(W_n = j) = P_\mu(X_n = j), \qquad P_\nu(Y_n = j) = \pi_j.$$

- $\sum_j |P_\mu(X_n = j) - \pi_j| \to 0$:

$$\text{LHS} = \sum_j |P(W_n = j) - P(Y_n = j)| \leqslant 2P_\nu(\tau > n) \to 0.$$

- 证明方法: 耦合(coupling).

例1.9.4. Wright-Fisher模型. N 个人, 表态: “1” 或 “0” . 每人每天独立任选一人并按图示随机跟随或改变. X_n = 第n 天表态“1” 的人数. 求: $EX_n = \sum_{j=0}^{N} jP(X_n = j)$ 在$n \to \infty$ 时的极限.



- 若$X_n = i$, 则$X_{n+1} \sim B(N, \rho_i)$,

  其中, $\rho_i = (1-u)\frac{i}{N} + v\frac{N-i}{N}$.

- 不可约、正常返、非周期, $EX_n \to \sum_{j=0}^{N} j\underline{\pi_j}$.
- $a = \sum_j j \underline{\sum_i \pi_i p_{ij}} = \sum_i \pi_i \sum_j j p_{ij} = \sum_i \pi_i N\rho_i$

  $= \sum_i \pi_i[(1-u)i + v(N-i)] = (1-u-v)a + vN$.

- $a = \frac{v}{u+v}N$.
- $u = v = \frac{1}{2}$. $\rho_i = \frac{1}{2}$. $X_1, X_2, \cdots$ i.i.d., $\pi = B(N, 1/2)$.

二、正常返、周期情形

定理 (定理1.9.6)

假设不可约. 则$\exists! d$ 以及S 的分割$D_0, D_1, \cdots, D_{d-1}$ 使得:

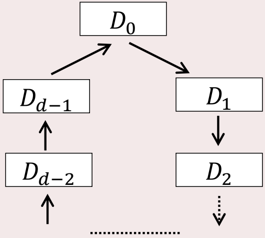
- $\forall\ r \geqslant 0,\ i \in D_r,\ s \geqslant 0,$

  $\sum_{j \in D_{r+s}} p_{ij}^{(s)} = 1;$

  注: 补充定义$D_{nd+r} := D_r$.

- $\forall\ r \geqslant 0,\ i, j \in D_r,\ \exists\ N$ 使得

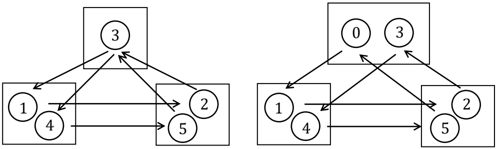
  $p_{ij}^{(nd)} > 0,\ \forall n \geqslant N.$



- 定理中的d 被称为“周期”(定义1.9.7). 非周期指$d = 1$.

- $\sum_{j\in D_{r+s}} p_{ij}^{(s)} = 1$;

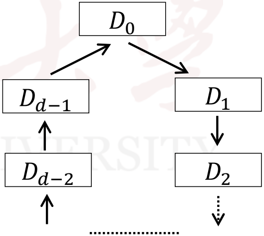
  $p_{ij}^{(nd)} > 0,\ \forall n \geqslant N$.



- 设$d > 1$. 令$\hat{\mathbf{P}} := \mathbf{P}^d$.

  则D_r's 是$\hat{\mathbf{P}}$ 的等价类, 且都是闭集.

- 在每个D_r 上, $\hat{\mathbf{P}}$ 不可约、非周期.



- $X_0 = i$, $\sigma_i = d\hat{\sigma}_i$. 故

  $\mathbf{P}$ 正(零、非)常返

  iff

  $\hat{\mathbf{P}}$ 正(零、非)常返.

- 若π 为$\mathbf{P}$ 的不变测度, 则

$$\pi(D_r) \equiv \frac{1}{d}, \quad \text{且 } d\pi|_{D_r} \text{ 是} \hat{\mathbf{P}} \text{ 的不变测度.}$$

三、零常返情形

命题 (命题1.9.9)

若不可约、零常返，则$p_{ij}^{(n)} \to 0, \forall i,j.$

- 不妨设非周期，否则考虑$\hat{\mathbf{P}} = \mathbf{P}^d$. 取定$i,j$. $\forall \varepsilon > 0$,
- Claim 1. $\exists M$ 使得$\forall n$, 在

$$p_{ij}^{(n)}, p_{ij}^{(n+1)}, \cdots, p_{ij}^{(n+M)}$$

中存在某个$p_{ij}^{(n+m)}$ 使得$p_{ij}^{(n+m)} < \varepsilon$.

- Claim 2. $\exists N$ 使得当$n \geqslant N$ 时,

$$|p_{ij}^{(n)} - p_{ij}^{(n+m)}| < \varepsilon, \quad \forall 1 \leqslant m \leqslant M.$$

- Claims $\Rightarrow p_{ij}^{(n)} < 2\varepsilon, \forall n \geqslant N.$ ✓

证明Claim 1. $\exists M$ 使得$\min\{p_{ij}^{(n)}, p_{ij}^{(n+1)}, \cdots, p_{ij}^{(n+M)}\} < \varepsilon, \forall n.$

- $E_i\sigma_i = P_i(\sigma_i \geqslant 1) + P_i(\sigma_i \geqslant 2) + \cdots = \sum_{i=0}^{\infty} P_i(\sigma_i > \ell) = \infty.$
  取M 使得$\sum_{\ell=0}^{M} P_i(\sigma_i > \ell) > \frac{1}{\varepsilon}.$
- 左边的事件是划分, 概率之和为1.

  $\{X_{n+M} = j\}$: $p_{ij}^{(n+M)} P_j(\sigma_j > 0)$

  $\{X_{n+M-1} = j, X_{n+M} \neq j\}$: $p_{ij}^{(n+M-1)} P_j(\sigma_j > 1)$

  $\{X_{n+M-2} = j, X_{n+M-1}, X_{n+M} \neq j\}$: $p_{ij}^{(n+M-2)} P_j(\sigma_j > 2)$

  $\ldots$

  $\{X_n = j, X_{n+1}, \cdots, X_{n+M} \neq j\}$: $p_{ij}^{(n)} P_j(\sigma_j > M)$

  $\{X_n, \cdots X_{n+M} \neq j\}$
- 反证法: 若$* \geqslant \varepsilon$, 则矛盾:

$$\sum \text{RHS} \geqslant \varepsilon \times * > 1.$$

证明, Claim 2. 当$n \gg 1$ 时, $|p_{ij}^{(n)} - p_{ij}^{(n+m)}| < \varepsilon, \forall 1 \leqslant m \leqslant M$.

- 设$\{Z_n = (W_n, Y_n)\}$ 的转移概率为$r_{(i,j)(k,\ell)} = p_{ik}p_{j\ell}$.
  $\mathbf{P}$ 非周期, 故$\{Z_n\}$ 不可约, (定理1.9.3).
- 情形I. $\{Z_n\}$ 非常返. 则

$$p_{ij}^{(n)} p_{ij}^{(n)} = \tilde{p}_{(i,i)(j,j)}^{(n)} \to 0.$$

$p_{ij}^{(n)} < \varepsilon, \forall n \geqslant N (\Rightarrow n + m \geqslant N)$.

- 情形II. $\{Z_n\}$ 常返. 取$\mu_k = \mathbf{1}_{\{k=i\}}, k \in S$;

$$\begin{aligned} |p_{ij}^{(n)} - p_{ij}^{(m+n)}| =& |P_{\mu \times \mu \mathbf{P}^m}(W_n = j) - P_{\mu \times \mu \mathbf{P}^m}(Y_n = j)| \\ \leqslant& 2P_{\mu \times \mu \mathbf{P}^m}(\tau > n) \to 0. \quad (\text{定理}1.9.3) \end{aligned}$$

四、总结

- 正常返、非周期:

$$p_{ij}^{(n)} \to \pi_j, \quad P_\mu(X_n = j) \to \pi_j.$$

  - 证: $\sum_{i\in D} \mu_i p_{ij}^{(n)} + \sum_{i\notin D} \mu_i p_{ij}^{(n)} \approx \sum_{i\in D} \mu_i \star \approx \sum_i \mu_i \star.$

- 正常返、周期:

$$p_{ij}^{(nd+s)} \to d\pi_j$$
$$p_{ij}^{(m)} \to 0, \quad m \neq nd+s,\ i \in D_r,\ j \in D_{r+s}.$$

- 零常返:

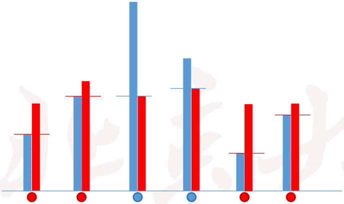
$$p_{ij}^{(n)} \to 0, \quad 但 \quad \sum_n p_{ij}^{(n)} = \infty.$$

- 非常返:

$$p_{ij}^{(n)} \to 0, \quad 且 \quad \sum_n p_{ij}^{(n)} < \infty.$$

§1.10 收敛速度

- 全变差距离: $d_{TV}(\mu,\nu) = \|\mu-\nu\| := \frac{1}{2}\sum_i |\mu_i - \nu_i|$.
- 注: 即$\frac{1}{2}\|\mu-\nu\|_{\ell^1}$.



- 注: 即$\mu(A)-\nu(A)$, $\nu(B)-\mu(B)$.
- $\|\mu\mathbf{P}-\nu\mathbf{P}\| \leqslant \|\mu-\nu\|$:

$$\text{LHS} = \frac{1}{2}\sum_j \left|\sum_i \mu_i p_{ij} - \sum_i \nu_i p_{ij}\right| \leqslant \frac{1}{2}\sum_j\sum_i |\mu_i - \nu_i| p_{ij} = \text{RHS}.$$

- 命题1.10.1(Dobrushin准则) 假设S 有限, $p_{ij} > 0, \forall i, j$. 则$\exists \alpha < 1$ 使得

$$d_{TV}(\mu\mathbf{P}, \nu\mathbf{P}) \leqslant \alpha d_{TV}(\mu, \nu), \quad \mu, \nu.$$

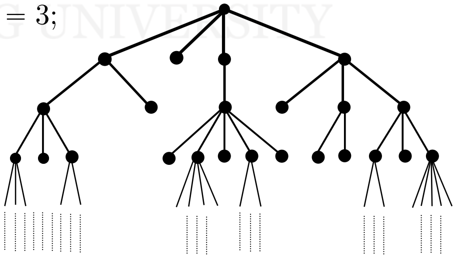
特别地, 若π 是$\mathbf{P}$ 的不变分布, 则$d_{TV}(\mu\mathbf{P}^n, \pi) \leqslant \alpha^n, \forall n \geqslant 0$.

- 命题1.10.2. 假设不可约、非周期, S 有限. 则$\exists C, \beta > 0$ 使得

$$d_{TV}(\mu\mathbf{P}^n, \pi) \leqslant Ce^{-\beta n}, \ \forall\ n \geqslant 0.$$

§1.11 分支过程(Branching process)

- $\{\xi_{ni} : n \geqslant 0; i \geqslant 1\}$ i.i.d.
  子代分布: $P(\xi = k) = p_k,\ k \in \mathbb{Z}_+$. (假设$p_1 \neq 1$)
- $X_0 = 1$, $X_1 = \xi_{01}$, $X_{n+1} = \xi_{n1} + \cdots + \xi_{nX_n}$.
  $X_n = 0 \Rightarrow X_{n+1} = 0$.
- 分支树$\mathbb{T}$ (GW 树, Galton-Watson tree):
  $\xi_{01} = 4$;
  $\xi_{11} = 2,\ \xi_{12} = 0,\ \xi_{13} = 1,\ \xi_{14} = 3$;
  $\xi_{21} = 3,\ \xi_{22} = 0,\ \xi_{23} = 5,$
  $\xi_{24} = 0,\ \xi_{25} = 2,\ \xi_{26} = 3$;
  $\xi_{31} = 3,\ \xi_{35} = 4,\ \xi_{3,13} = 5,$
  $\xi_{3i} = 2,\ i = 3, 7, 11, \cdots\cdots$



- 灭绝时间: $\tau_0 = \inf\{n \geqslant 0 : X_n = 0\}$.
- 灭绝概率: $\rho = P_1(\tau_0 < \infty)$.
- 子代分布的母函数: $f(s) = f_\xi(s) = Es^\xi = \sum_k P(\xi = k)s^k$.
- $0 \leqslant s \leqslant 1 \Rightarrow 0 \leqslant f(s) \leqslant 1$.
- $f''(s) \geqslant 0$.
- $m = E\xi = \sum_k kp_k = \lim_{s\uparrow 1} f'(s) \triangleq f'(1)$.

- 命题1.11.1. $f_{X_n}(s) = f^{(n)}(s)$.
- 证: $$\begin{aligned} f_{X_{n+1}}(s) &= Es^{X_{n+1}} \\ &= \sum_{k=0}^{\infty} P(X_n = k) E(s^{\xi_1+\cdots+\xi_k} | X_n = k) \\ &= \sum_{k=0}^{\infty} P(X_n = k) f(s)^k = f_{X_n}(f(s)). \end{aligned}$$
- 命题1.11.2. $EX_n = m^n$. 若$m < 1$, 则$\rho = 1$.
- 证: 用归纳法. $EX_0 = 1 = m^0$,

$$\begin{aligned} EX_{n+1} &= E\sum_{j=1}^{X_n} \xi_{nj} = \sum_{k=0}^{\infty} P(X_n = k) E\left(\sum_{j=1}^{k} \xi_{nj} \Big| X_n = k\right) \\ &= \sum_{k=0}^{\infty} P(X_n = k) k E\xi = mEX_n. \end{aligned}$$

- $P(\tau_0 = \infty) \leqslant P(\tau_0 > n) = P(X_n \geqslant 1) \leqslant EX_n = m^n \to 0$.

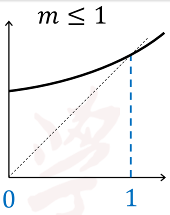
命题 (命题1.11.3)

灭绝概率$\rho = P_1(\tau_0 < \infty)$ 是方程$s = f(s)$ 的最小非负解.

于是$\rho < 1 \Leftrightarrow m = \sum_k k p_k > 1$.

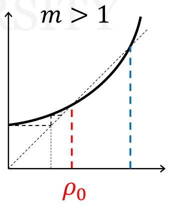
- $\rho = f(\rho)$:

$$\rho = \sum_{k=0}^{\infty} P(\tau_0 < \infty, X_1 = k)$$
$$= p_0 + \sum_{k=1}^{\infty} p_k \rho^k = f(\rho).$$



- $\rho = \rho_0$:

$$\rho = P(\tau_0 < \infty) = \lim_{n\to\infty} P(\tau_0 \leqslant n)$$
$$= \lim_{n\to\infty} P_1(X_n = 0) = \lim_{n\to\infty} f_{X_n}(0)$$
$$= \lim_{n\to\infty} f^{(n)}(0) \leqslant \lim_{n\to\infty} f^{(n)}(\rho_0) = \rho_0.$$



- 相变: $m=\sum_k kp_k$, $\rho=P_1(\tau_0<\infty)$.

参数范围	$m<1$	$m=1$	$m>1$
灭绝概率	$\rho=1$	$\rho=1$	$\rho<1$
名称	次/下临界	临界	超/上临界

- §1.11 习题1. 除状态0 外, 均为暂态.
- $\rho=P(\lim_{n\to\infty}X_n=0)$.
- $1-\rho=P(\lim_{n\to\infty}X_n=\infty)$.

例1.11.6. 子代分布列: $p_k = p(1-p)^k$, $k = 0, 1, 2, \cdots$.
其中, $0 < p < 1$. 求灭绝概率ρ.

- 先求$m = E\xi$:

$$m = \sum_{k=0}^{\infty} kp_k = \sum_{k=0}^{\infty} kp(1-p)^k = \frac{p(1-p)}{(1-(1-p))^2} = \frac{1-p}{p}.$$

- 若$p \geqslant 1/2$, 则$m \leqslant 1$, 故$\rho = 1$.
- 若$p < 1/2$, 则$m > 1$. 解方程

$$\rho = f(\rho) = \sum_k \rho^k p_k = \sum_k \rho^k p(1-p)^k = \frac{p}{1-\rho(1-p)}.$$

得$\rho = p/(1-p)$ 或1(舍弃).