第三章、布朗运动

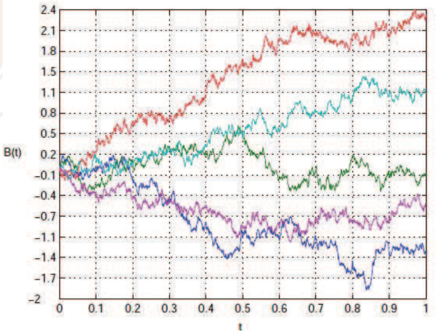
§3.1 定义

1. 定义

X_t = 时间t 之后的位移. 噪声.

(A1) X_t 各向同性.   (A3) X_t 关于t 连续.

(A2) $X_{t+s} - X_t$ 与X_t 相互独立, 与X_s 具有相同的分布.



图片来源:“Brownian motion”, P. Mörters & Y. Peres.

定义 (定义3.1.1)

假设$B_0=0$. 若$\forall t\geqslant 0,\ s>0,\ n\geqslant 2,\ \forall 0<t_1<\cdots<t_n$.

(B1) $B_{t+s}-B_t\sim N(0,s)$,

(B2) $B_{t_1}, B_{t_2}-B_{t_1},\ldots,B_{t_n}-B_{t_{n-1}}$ 相互独立,

(B3) 轨道连续: $\forall\omega,\ B_t(\omega)$ 关于t 连续.

则称$\{B_t:t\geqslant 0\}$ 为标准布朗运动(Brownian motion).

- $X_t=x+\sigma B_t,\ P_x,\ P_\mu$.
- (B1) & (B2). 任意有限维边缘分布: $\mathcal{L}(B_{t_1},\ldots,B_{t_n})$.
- (B3). 轨道: 给定ω, 视为t 的函数, 性质.

(B1) $B_{t+s} - B_t \overset{d}{=} B_s \sim N(0, s)$,

(B2) $B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}$ 相互独立,

- (B) 独立平稳增量过程.
- 独立平稳增量, 可实现(B3)$'$ 轨道左连右极.
- **PP**(λ): $\sim \mathcal{P}(\lambda s)$. (B) 成立, 但(B3) 不成立.
- (B) & (B3) 成立$\Rightarrow \sim N(s\mu, s\sigma^2)$.
- "版本": 同分布. 例, (B1) & (B2) 成立, 但(B3) 不成立.

(B1) & (B2). 任意有限维联合分布$\mathcal{L}(B_{t_1}, \ldots, B_{t_n})$:

- $N(0, s)$ 的密度:

$$p_s(z) = \frac{1}{\sqrt{2\pi s}} e^{-\frac{z^2}{2s}}.$$

- $(B_{t_1}, \ldots, B_{t_n})$ 的联合密度: 记$\Delta t_k = t_k - t_{k-1}$,

$$p_{\vec{t}}(\vec{x}) = p_{t_1,\cdots,t_n}(x_1, \cdots, x_n) = \prod_{k=1}^{n} p_{\Delta t_k}(x_k - x_{k-1}).$$

  - $\vec{\xi} = (\xi_1, \ldots, \xi_n) = (B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$:

$$p_{\vec{\xi}}(\vec{y}) = \prod_{k=1}^{n} p_{\Delta t_k}(y_k), \quad p_{\vec{t}}(\vec{x}) = p_{\vec{\xi}}(\vec{y}), \; y_k = \Delta x_k.$$

- 转移(概率)密度: $(0 < t_1 < \ldots < t_n < t < t + s)$

$$p_{t+s|t;t_1,t_2,\cdots,t_n}(y|x; x_1, x_2, \cdots, x_n) = p_s(y - x) =: p_s(x, y).$$

2. 转移密度. $p_t(x,y)=p_t(y-x)=\frac{1}{\sqrt{2\pi t}}e^{-\frac{(y-x)^2}{2t}}$.

- 格林函数$G(x,y)=\int_0^\infty \frac{1}{\sqrt{2\pi}\cdot\sqrt{t}}e^{-\frac{(y-x)^2}{2t}}dt=\infty$.
- $p_{t+s}(x,y)=\int_{\mathbb{R}} p_t(x,z)p_s(z,y)dz$.
- Kolmogorov 前进、后退方程:

$$\frac{\partial p_t(x,y)}{\partial t}=\frac{1}{2}\frac{\partial^2 p_t(x,y)}{\partial y^2}=\frac{1}{2}\frac{\partial^2 p_t(x,y)}{\partial x^2},$$

- $f:S=\mathbb{R}\to\mathbb{R}$, $f_t(x):=E_xf(X_t)=Ef(x+B_t)$.

$$\frac{\partial}{\partial t}f_t(x)=\frac{1}{2}\frac{\partial^2}{\partial x^2}f_t(x)=(\mathcal{L}f_t)(x).$$

- $\mathcal{L}:f\mapsto \mathcal{L}f=g,\quad g(x)=\frac{1}{2}f''(x)$.
- $f_t(x)=Ef(x+B_t)=f(x)+\frac{1}{2}f''(x)t+o(t)$,
  $f(x+B_t)=f(x)+f'(x)B_t+\frac{1}{2}f''(x)B_t^2+\star\star$.

3. 高斯过程

- 正态/高斯分布: $\vec{X} := (X_1, \cdots, X_n)^T \sim N(\vec{\mu}, \Sigma)$

$$\rho_{\vec{X}}(\vec{x}) = \frac{1}{\sqrt{(2\pi)^n|\Sigma|}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})\cdot\Sigma(\vec{x}-\vec{\mu})}.$$

$$Ee^{i\vec{t}\cdot\vec{X}} = e^{i\vec{t}\cdot\vec{\mu}-\frac{1}{2}\vec{t}\cdot(\Sigma\vec{t})}.$$

- 期望: $\forall t \geqslant 0,\ EB_t = 0.$

  协方差: $\forall 0 < t < s,\ EB_tB_s = EB_t^2 + EB_t(B_s - B_t) = t.$

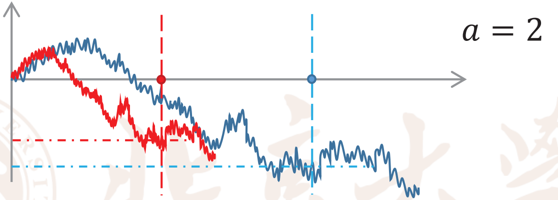
- 等价定义: (C1) & (C2) & (B3).

  (C1) $(B_{t_1}, \dots, B_{t_n})$ 服从联合正态/高斯.

  (C2) $EB_t = 0, \quad EB_tB_s = t \wedge s.$

时空尺度: 时间= 空间^2. $(t = EB_t^2)$

- $\forall a > 0$, $X_t = \frac{1}{\sqrt{a}} B_{at}$, $t \geqslant 0$ 是标准布朗运动.



  - (C1) $(X_{t_1}, \ldots, X_{t_n}) = \frac{1}{\sqrt{a}}(B_{at_1}, \ldots, B_{at_n})$ 服从联合正态/高斯.
  - (C2) $EX_t = 0$, $EX_tX_s = \frac{1}{a}EB_{at}B_{as} = \frac{1}{a}(at \wedge as) = t \wedge s$.
  - (B3) 轨道连续.
- 自相似、分形.

4. 构造.

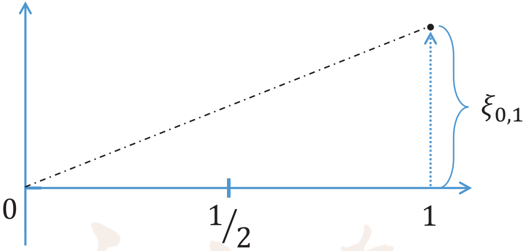
$\{B_t, 0 \leqslant t \leqslant 1\}$ 的Levi 构造:

- 思想: 若$X \sim N(0,\sigma^2)$, 则
$$X = Y \bigoplus Z, \text{其中} Y, Z \overset{\text{i.i.d.}}{\sim} N(0, \tfrac{1}{2}\sigma^2).$$
- 实现: 取$\tilde{X}$ 与X i.i.d., 令

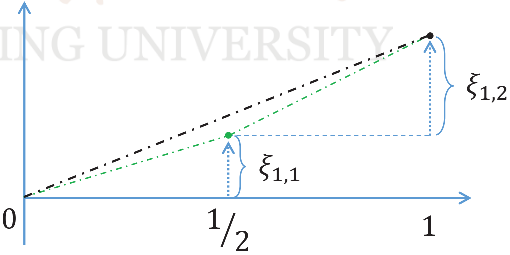
$$Y := \frac{1}{2}(X + \tilde{X}) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(X + \tilde{X}),$$

$$Z := \frac{1}{2}(X - \tilde{X}) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(X - \tilde{X}).$$

- 任取$\xi = \xi_{0,1} \sim N(0,1)$.
  令$B_1 = \xi_{0,1}$.
  线性插值得到$B_t^{(0)}$.



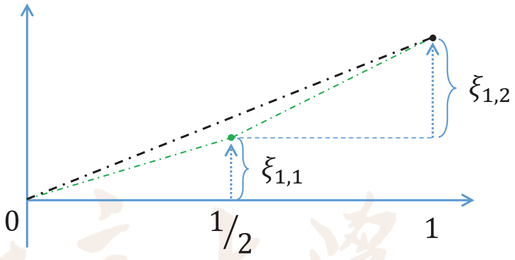
- 取$\tilde{\xi} = \tilde{\xi}_{0,1}$. 令
  $\xi_{1,1} = \frac{1}{2}(\xi + \tilde{\xi})$,
  $\xi_{1,2} = \frac{1}{2}(\xi - \tilde{\xi})$. 于是$\xi = \xi_{1,1} \bigoplus \xi_{12}$. $B_{\frac{1}{2}} = \xi_{1,1}$.
- 令$B_{\frac{1}{2}} = \xi_{1,1} = \frac{1}{2}(\xi + \tilde{\xi})$.
  插值得到$B_t^{(1)}$.
- D_1:
  $= \max\limits_{0 \leqslant t \leqslant 1} |B_t^{(1)} - B_t^{(0)}|$
  $= |\frac{1}{2}\xi - \frac{1}{2}(\xi + \tilde{\xi})|$
  $= \frac{1}{2}|\tilde{\xi}|$.



- 取$\tilde{\xi}_{1,1}, \tilde{\xi}_{1,2} \sim N(0, \frac{1}{2})$，使得

  $\xi_{11} = \xi_{2,1} \bigoplus \xi_{2,2}$,

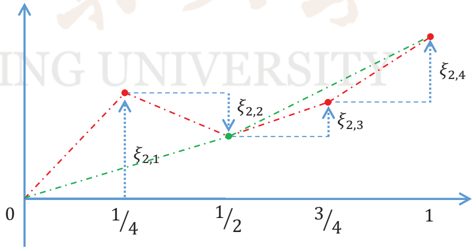
  $\xi_{1,2} = \xi_{2,3} \bigoplus \xi_{2,4}$.



- 获得$B_{\frac{1}{4}}$, $B_{\frac{1}{2}}$, $B_{\frac{3}{4}}$.
  插值得到$B_t^{(2)}$.

- D_2:

  $$= \max_{0 \leqslant t \leqslant 1} |B_t^{(2)} - B_t^{(1)}| = \frac{1}{2} \max\{|\tilde{\xi}_{1,1}|, |\tilde{\xi}_{1,2}|\}.$$

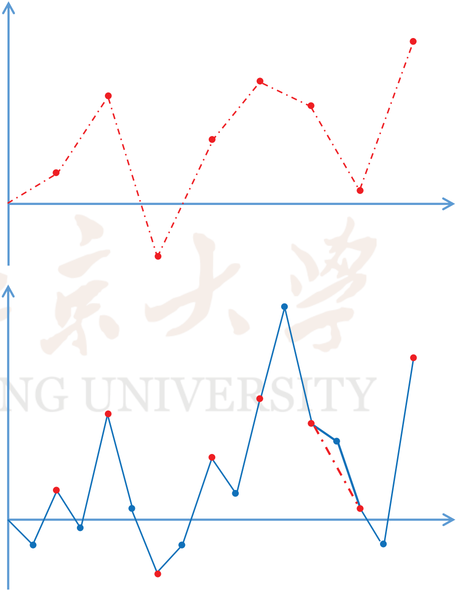


- 假设已有$B_t^{(n)}$.

- 取$\tilde{\xi}_{n,i} \sim N(0, \frac{1}{2^n})$,
  获得$B_{\frac{i}{2^{n+1}}}$,
  插值得到$B_t^{(n+1)}$.

- D_n:
  $$= \max_{0\leqslant t\leqslant 1} |B_t^{(n+1)} - B_t^{(n)}|$$
  $$= \frac{1}{2} \max_{1\leqslant i\leqslant 2^n} |\tilde{\xi}_{n,i}|.$$

- 往证$D_n = O(2^{-n/4}\sqrt{n})$.



- $A_n := \{\max_{1\leqslant i\leqslant 2^n} |\tilde{\xi}_{n,i}| \geqslant 2^{-n/4}\sqrt{n}\}$, $n \geqslant 1$ 相互独立.
- $P(A_n) \leqslant 2^n P(\sqrt{2^{-n}}|Z| \geqslant 2^{-n/4}\sqrt{n})$

$$= 2^n P(|Z| \geqslant 2^{n/4}\sqrt{n}) \leqslant 2^n \frac{EZ^4}{2^n n^2} = \frac{1}{n^2} EZ^4.$$

$\Rightarrow \sum_n P(A_n) < \infty$.

- Borel-Cantelli引理: $P(A_n\text{发生有限次}) = 1$.

$\forall \omega$, $\exists N(\omega)$ s.t. $\forall n \geqslant N(\omega)$, $\omega \notin A_n$, 即

$$\max_{0\leqslant t\leqslant 1} |B_t^{(n+1)}(\omega) - B_t^{(n)}(\omega)| \leqslant 2^{-n/4}\sqrt{n}.$$

- $B_t^{(n)}(\omega)$, $t \in [0,1]$ 一致收敛到某连续函数$B_t(\omega)$, $t \in [0,1]$.
- 补充定义$B_t(\omega) \equiv 0$, 其他ω.

5. d 维标准布朗运动: $\vec{B}_t = (B_t^{(1)}, \ldots, B_t^{(d)})^T$.

- $\{B_t^{(i)}, t \geqslant 0\}$, $i = 1, \ldots, d$ 是i.i.d. 一维布朗运动.
  - (D1) $\{B_t^{(i)}; i, t\}$ 中任意有限个服从联合正态,
  - (D2) $EB_t^{(i)} = 0$, $EB_t^{(i)} B_s^{(j)} = 1_{\{i=j\}} \cdot t \wedge s$,
  - (D3) 轨道连续.
- 各向同性(命题3.1.7): 假设$\mathbf{O}$ 是d 维正交矩阵,
  则$\vec{X}_t = \mathbf{O}\vec{B}_t$ 仍然是d 维标准布朗运动.
  - (D1) $\surd$. (D3) $\surd$. (D2): $EX_t^{(i)} = 0$, $\surd$.
  - $EX_t^{(i)} X_s^{(j)} = E\underline{\sum_k o_{ik} B_t^{(k)}} \sum_\ell o_{j\ell} B_s^{(\ell)}$

    $= \sum_{k,\ell} o_{ik} o_{j\ell} 1_{\{k=\ell\}} \cdot t \wedge s = \sum_k o_{ik} o_{jk} t \wedge s = 1_{\{i=j\}} \cdot t \wedge s.$

- 转移密度: $p_t(\vec{x},\vec{y}) = \frac{1}{\sqrt{(2\pi t)^d}} e^{-\frac{\|\vec{y}-\vec{x}\|^2}{2t}}$.

$$\vec{B}_{s+t} - \vec{B}_s \sim N(\vec{0}, t\cdot\mathbf{I}).$$

- $p_{t+s}(\vec{x},\vec{y}) = \int_{\mathbb{R}^d} p_t(\vec{x},\vec{z})p_s(\vec{z},\vec{y})d\vec{z}$.
- Kolmogorov 前进、后退方程:

$$\frac{\partial p_t(\vec{x},\vec{y})}{\partial t} = \frac{1}{2}\Delta_y p_t(x,y) = \frac{1}{2}\Delta_x p_t(x,y),$$

$\Delta_x = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}$.

- 格林函数: $G(\vec{x},\vec{y}) = \int_0^\infty p_t(\vec{x},\vec{y})dt$.
  - $d=2$: $G(\vec{x},\vec{y}) = \int_0^\infty \frac{1}{2\pi\cdot t} e^{-\frac{\|\vec{y}-\vec{x}\|^2}{2t}} dt = \infty$.
  - $d\geqslant 3$: $G(\vec{x},\vec{y}) = \int_0^\infty \frac{1}{2\pi\cdot t^{d/2}} e^{-\frac{\|\vec{y}-\vec{x}\|^2}{2t}} dt < \infty$.

6. 可测性.

- 样本空间$(\Omega,\mathcal{F},P)$, 过程$\{B_t\}$, 满足(B1), (B2), (B3).
- 轨道空间: $\hat{\Omega}=\{\hat{\omega}:[0,\infty)\to\mathbb{R}\}$, 坐标过程: $\{\hat{\omega}_t\}$.
- $\hat{\mathcal{F}}=\sigma(\{\hat{\omega}_t\leqslant x\}:\forall t,\forall x)$.
- $\mathbf{B}:(\Omega,\mathcal{F})\to(\hat{\Omega},\hat{\mathcal{F}})$, $\omega\mapsto\hat{\omega}_t=B_t(\omega)$, 可测映射:
  $\mathbf{B}^{-1}\{\hat{\omega}_t\leqslant x\}=\{B_t(\omega)\leqslant x\}\in\mathcal{F}$.
- 诱导$\hat{P}$: $\hat{P}(\hat{\omega}_{t_1}\leqslant x_1,\ldots,\hat{\omega}_{t_n}\leqslant x_n)$
  $=P(B_{t_1}\leqslant x_1,\ldots,B_{t_n}\leqslant x_n)$. (B1) & (B2).
- $C[0,\infty)\notin\hat{\mathcal{F}}$.
- 样本轨道空间: ⋆⋆ & (B3).
  $\tilde{\Omega}=C[0,\infty)$, $\tilde{\mathcal{F}}=\{A\cap\tilde{\Omega},A\in\mathcal{F}\}$, $\tilde{P}(A\cap\tilde{\Omega})=P(A)$.

§3.2 不变原理(Invariant Principle)

依分布收敛.

- 分布、分布函数: $F_X(x) = P(X \leqslant x)$.
- 例: Bernoulli 投币. $\xi_n \overset{d}{=} \xi_1$, 但$\xi_n = \xi_1$ 不成立.
- 如果对任意F_X 的连续点x 都有$P(Y_n \leqslant x) \to P(X \leqslant x)$, 那么称$Y_n$ 依分布收敛于X, 记为$Y_n \xrightarrow{d} X$.
- 一般性定义.

  - $\mathbb{C}$: (完备可分)距离空间.
  - Y_n, $\forall n \geqslant 1$, X: 取值于$\mathbb{C}$ 的随机变量.
  - $Y_n \xrightarrow{d} X$: 若$P(X \in \partial B) = 0$, 则$P(Y_n \in B) \to P(X \in B)$.
  - $Y_n \xrightarrow{d} X$: $\forall \mathbb{C}$ 上有界连续函数f, $Ef(Y_n) \to Ef(X)$.

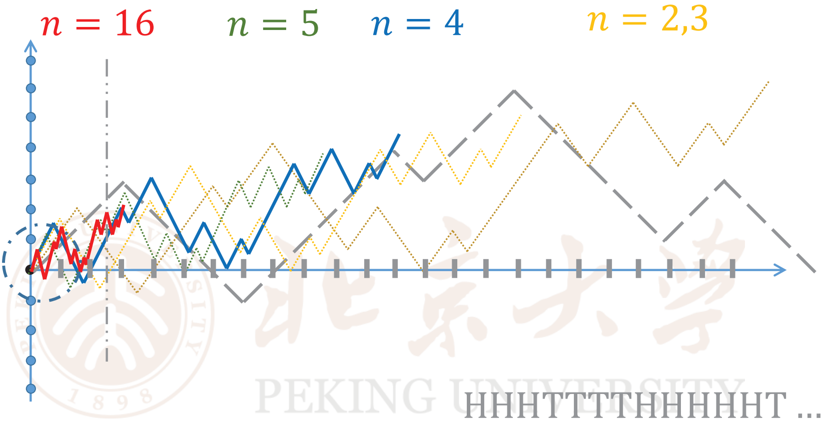
- CLT: $\xi_1, \xi_2, \cdots$ i.i.d., $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$, $S_m = \xi_1 + \cdots + \xi_m$. 则

$$\frac{1}{\sqrt{n}} S_n \xrightarrow{d} Z \sim N(0,1).$$

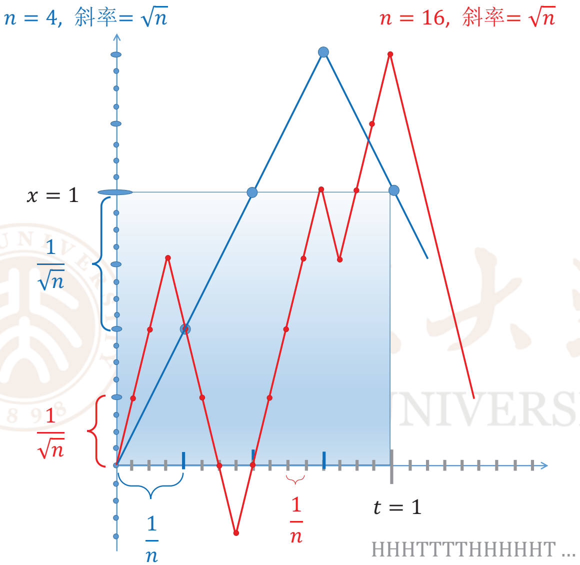
- 时间 = 空间^2.
- $\Delta t = \frac{1}{n}$, $\Delta x = \frac{1}{\sqrt{n}}$. $\frac{1}{\sqrt{n}} S_n = \frac{1}{\sqrt{n}} \xi_1 + \ldots + \frac{1}{\sqrt{n}} \xi_n \overset{d}{\approx} B_1$.
- $t = m\Delta t$, 即$m = nt$,

$$\frac{1}{\sqrt{n}} \xi_1 + \ldots + \frac{1}{\sqrt{n}} \xi_m = \sqrt{t} \cdot \frac{1}{\sqrt{nt}} S_{nt} \xrightarrow{d} \sqrt{t} Z \overset{d}{=} B_t.$$

- $\{\frac{1}{\sqrt{n}} S_{nt} : t \geqslant 0\}$ 独立平稳增量, 且增量的分布 $\overset{n\to\infty}{\longrightarrow} N(0, s)$.
- 插值: $\forall t \in [m, m+1]$, 令$S_t = S_m + (t-m)(S_{m+1} - S_m)$.



- $\frac{1}{\sqrt{n}}S_n \xrightarrow{d} Z \sim N(0,1)$, 但是a.s.-$\lim\limits_{n} \frac{1}{\sqrt{n}}S_n$ 不存在.
- §3.1 习题8. $\{B_{T-t} - B_T : 0 \leqslant t \leqslant T\}$, $\{-B_t : t \geqslant 0\}$ 为布朗运动、反射原理.



- 轨道处处不可微、$[0,\varepsilon)$ 中含无穷多个零点.

- $\mathbb{R}$ 中的连续轨道: $\varphi \in C[0,\infty)$.
- 布朗运动: 产生一条随机的连续轨道X, $X(t) = B_t$.
- 投币: $\{S_n\}$. 插值产生一条随机的连续轨道Y,
- 尺度变换: $\Psi_n : C[0,\infty) \hookleftarrow$, $\underset{\sim}{\Psi_n(\psi)}(t) = \frac{1}{\sqrt{n}}\psi(nt)$.

  产生$Y_n := \Psi_n(Y)$.
- "不变原理": $Y_n \xrightarrow{d} X$. (注: 不是$Y_n \xrightarrow{a.s.} Y \overset{d}{=} X$.)
  - $C[0,\infty)$ 上的距离?

- 完备可分距离空间: 给定$T>0$, (取$\mathbb{C}=C[0,T]$), 令

$$d_T(\varphi,\psi):=\max_{0\leqslant t\leqslant T}|\varphi(t)-\psi(t)|,\quad \forall\varphi,\psi\in C[0,T].$$

则$(C[0,T],d_T)$ 是一个完备可分的距离空间.

- 截断: $\Phi_T: C[0,\infty)\to C[0,T]$, $\varphi\to\varphi|_{[0,T]}$.
- 不变原理: $\Phi_T(Y_n)\xrightarrow{d}\Phi_T(X)$, $\forall T>0$.

定理 (不变原理, 定理3.2.1)

$\forall T>0$, $\forall C[0,T]$ 上的任意有界连续泛函 f,

$$\lim_{n\to\infty} Ef\Big(\{\frac{1}{\sqrt{n}}S_{nt}:0\leqslant t\leqslant T\}\Big)=Ef\Big(\{B_t:0\leqslant t\leqslant T\}\Big).$$

例: $\frac{1}{\sqrt{n}}\max_{0\leqslant m\leqslant n} S_m \xrightarrow{d} \max_{0\leqslant t\leqslant 1} B_t$.

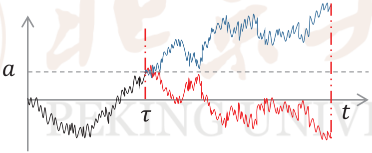
- 令$F: C([0,1]) \to \mathbb{R}$, $\varphi \mapsto \max_{0\leqslant t\leqslant 1}\varphi(t)$. 则$F$ 是连续函数.
  - $F\big(\{\frac{1}{\sqrt{n}}S_{nt} : 0\leqslant t\leqslant 1\}\big) = \frac{1}{\sqrt{n}}\max_{0\leqslant m\leqslant n} S_m$.
  - 若$d_T(\varphi,\psi) := \max_{0\leqslant t\leqslant T}|\varphi(t)-\psi(t)| \leqslant \varepsilon$,
    则$F(\varphi) = \varphi(t_\varphi) \leqslant \psi(t_\varphi) + \varepsilon \leqslant \psi_{t_\psi} + \varepsilon = F(\psi) + \varepsilon$.
    故, $|F(\varphi) - F(\psi)| \leqslant \varepsilon$.
  - F 不是有界(连续)函数!
- $\forall\mathbb{R}$ 上有界连续函数f,
  $Ef\circ F\big(\{\frac{1}{\sqrt{n}}S_{nt} : 0\leqslant t\leqslant 1\}\big) \to Ef\circ F\big(\{B_t : 0\leqslant t\leqslant 1\}\big)$.
  即, $\frac{1}{\sqrt{n}}\max_{0\leqslant m\leqslant n} S_m \xrightarrow{d} \max_{0\leqslant t\leqslant 1} B_t$.

§3.3 轨道性质

- $(\Omega, \mathcal{F}, P)$: $(\tilde{\Omega} = C[0,\infty), \tilde{F}, \tilde{P})$.
- 首达时: $\tau = \tau_a = \inf\{t \geqslant 0 : B_t = a\}$.
- 最大值: $M_t = \max_{0\leqslant s\leqslant t} B_s$.
- 零点: $\mathcal{Z} = \{t \geqslant 0 : B_t = 0\}$.
- 轨道处处不可微.

1. 首达时.

- 定理3.3.1(强马氏性). 在$\tau = \tau_a < \infty$ 的条件下, $\{W_t := B_{\tau+t} - B_\tau, t \geqslant 0\}$ 为标准布朗运动.
- 推论: $\hat{B}_t := \begin{cases} B_t, & t \leqslant \tau_a; \\ 2a - B_t, & t > \tau_a. \end{cases}$ 则$\{\hat{B}_t\}$ 是标准布朗运动.



- 命题3.3.2(反射原理).

$$P_0(\tau_a \leqslant t) = 2P_0(B_t > a), \quad \forall a > 0.$$

- $P_0(\star, B_t > a) = P_0(\star, \hat{B}_t < a) = P_0(\star, B_t < a)$.

$$P_0(\tau_a \leqslant t) = 2P_0(B_t > a) = P_0(|B_t| > a),\ \forall a > 0.$$

- 常返: $P_0(\tau_a < \infty) = 1.$

$$P_0(\tau_a \leqslant t) = P_0(|B_t| > a) = P\left(|Z| > \frac{|a|}{\sqrt{t}}\right) \to 1.$$

- $P_0(\forall n \geqslant 1, \exists t \geqslant n \text{ s.t. } B_t = 0) = 1$:

$$P_0(A_n) = \int p_n(x) P_x(\tau_0 < \infty) dx = 1, \quad \forall n \geqslant 1.$$

- $\sigma := \inf\{t \geqslant \tau_1 : B_t = 0\} \stackrel{d}{=} \tau_1 + \hat{\tau}_1$. $\sigma_1, \sigma_2, \ldots$ i.i.d.,

$$T_n = \sigma_1 + \ldots + \sigma_n \to \infty, \quad B_{T_n} = 0,\ \forall n.$$

$$P_0(\tau_a \leqslant t) = 2P_0(B_t > a) = P_0(|B_t| > a),\ \forall a > 0.$$

- 零常返:

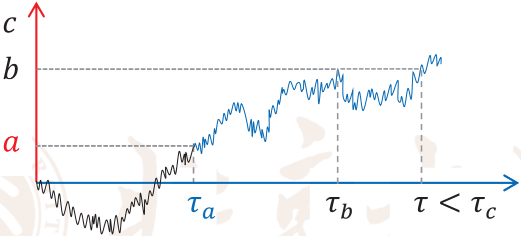
$$P_0(\tau_a > t) = P\left(|Z| \leqslant \frac{a}{\sqrt{t}}\right) = O(\frac{1}{\sqrt{t}})$$

$$\Rightarrow \quad E_0\tau_a = \int_0^\infty P_0(\tau_a > t)dt = \infty.$$

- dx 为不变测度.
- 推论3.3.3(τ_a 是连续型).

$$\rho_{\tau_a}(t) = 2p_Z\left(\frac{a}{\sqrt{t}}\right)\frac{1}{2}\frac{a}{\sqrt{t^3}} = \frac{a}{\sqrt{2\pi t^3}}e^{-\frac{a^2}{2t}}, \quad t > 0.$$

- $\{\tau_a, a \geqslant 0\}$: 独立平稳增量, 但不满足(B3).



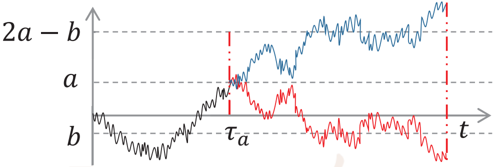
- 令 $\tilde{B}_t = \begin{cases} B_t, & t \neq \tau_a; \\ 0, & t = \tau_a, \end{cases}$ 则 $P(\tilde{B}_t \neq B_t) = 0, \forall t.$

  $\{\tilde{B}_t\}$ 满足(B1) & (B2), 但并不满足(B3).

- $\{\hat{B}_t = \frac{1}{c} B_{c^2 t}\} \Rightarrow \hat{\tau}_a = \frac{1}{c^2} \tau_{ca}.$

2. 最大值$M_t = \max_{0\leqslant s\leqslant t} B_s$.

- 命题3.3.5. $M_t \overset{d}{=} |B_t|$.

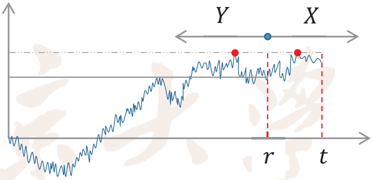


- $M_t > a$ iff $\tau_a \leqslant t$,

  $P(\tau_a \leqslant t) = 2P_0(B_t > a) = P_0(|B_t| > a)$.
- 习题3. (M_t, B_t) 是连续型.
- $M_t \leqslant a, B_t \leqslant b \longleftrightarrow M_t > a, B_t \leqslant b$, $\forall a > b \vee 0$.
- $P_0(A) = P_0(\tau_a \leqslant t, B_t > 2a - b) = P_0(B_t > 2a - b)$.
- 习题5. $P_0(M_t > a | B_t = M_t)$: $X_t := M_t - B_t = 0$.

推论3.3.6(最大值点唯一). $\forall t > 0$,

$$P_0(\exists\ t_1 < t_2 < t,\ \text{s.t. } B_{t_1} = B_{t_2} = M_t) = 0.$$



- $\exists r \in \mathbb{Q}_+$ 使得$t_1 < r < t_2$.

$$\star \leqslant \sum_{r\in\mathbb{Q}_+} P_0(\exists\ t_1 < r, r < t_2 < t,\ \text{s.t. } B_{t_1} = B_{t_2} = M_t).$$

- $X_s = B_{s+r} - B_r,\ s \geqslant 0;\ Y_s = B_{r-s} - B_r,\ 0 \leqslant s \leqslant r.$

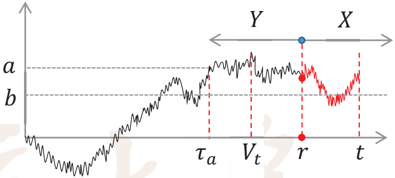
$$\star \leqslant P_0\Big(\max_{0\leqslant s\leqslant r} Y_s = \max_{0\leqslant s\leqslant t-r} X_s\Big) = 0.$$

最大值点$V_t \in [0,t]$.

- $M_t > a, V_t \leqslant r$

  iff $\tau_a \leqslant r$ 且

  $M_{t-r}^{(X)} < M_r - B_r$.



- iff $M_r > a$, $M_{t-r}^{(X)} < M_r - B_r$.

$$P_0(A) = \iint \rho_{(M_r,B_r)}(x,y) 1_{\{x>a\}} P_0(M_{t-r}^{(X)} < x-y) dxdy.$$

- $\forall a > b \vee 0,\ M_t > a, V_t \leqslant r, B_t \leqslant b$ iff A, $B_{t-r}^{(X)} + B_r \leqslant b$.

$$P_0(A,B) = \iint \rho_{(M_r,B_r)}(x,y) 1_{\{x>a\}}$$

$$\times P_0(M_{t-r}^{(X)} < x-y, B_{t-r}^{(X)} \leqslant b-y) dxdy.$$

推论3.3.7. $P_0\left(\lim_{t\to\infty}\frac{B_t}{t}=0\right)=1$.

- SLLN: $\frac{B_n}{n}\to 0$.
- 令$D_n=\max_{n\leqslant s\leqslant n+1}|B_s-B_n|$.
  则$D_1, D_2, \ldots$ i.i.d., $ED_n\leqslant 2EM_1=2E|B_1|$.
- SLLN/Borel-Cantelli引理: $\frac{1}{n}D_n=\frac{1}{n}S_n-\frac{n-1}{n}\cdot\frac{1}{n-1}S_n\xrightarrow{\text{a.s.}}0$.
- 重对数律(LIL):

$$P_0\left(\limsup_{t\to\infty}\frac{B_t}{\sqrt{2t\log\log t}}=1\right)=1.$$

3. 轨道的光滑性.

- 命题3.3.8. $P_0(\exists 0 \leqslant a < b \text{使得} B|_{(a,b)} \text{ 单调}) = 0.$

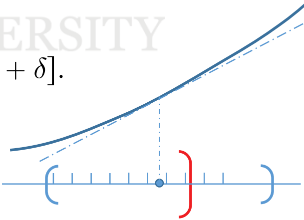
$$P_0(A) \leqslant \sum_{r,s\in Q} P_0(A_{r,s}), \quad P(A_{r,s}) \leqslant 2P(Z > 0)^n \to 0.$$

- 命题3.3.9. 轨道处处不可微.
- 数学分析: φ 为$[0,1]$ 上(给定的)函数.
- $\varphi'(t_0)$ $\exists$: $\exists M > 0$, $\exists \delta > 0$, 使得

$$|\varphi(t) - \varphi(t_0)| \leqslant M|t - t_0|, \ \forall\ t \in [t_0 - \delta, t_0 + \delta].$$

- $\forall n > \frac{4}{\delta}$, $\exists 0 \leqslant k \leqslant n-3$ s.t.

$$\left|\varphi\left(\frac{k+i}{n}\right) - \varphi\left(\frac{k+i-1}{n}\right)\right| \leqslant \frac{8M}{n}, \ i = 1, 2, 3.$$



- $\{B_t$在 $[0,1]$ 中有可微点$\} \subseteq \bigcup_{M=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} \bigcup_{k=0}^{n-3} A_{n,k}$,

$$A_{n,k} = \left\{ \left| B(\frac{k+i}{n}) - B(\frac{k+i-1}{n}) \right| \leqslant \frac{8M}{n},\ i = 1,2,3 \right\}.$$

- 往证$\lim_{n\to\infty} \sum_{k=0}^{n-3} P_0(A_{n,k}) = 0$.

$$P_0(A_{n,k}) = P\left(\frac{1}{\sqrt{n}}|Z| \leqslant \frac{8M}{n} \right)^3 = P\left(|Z| \leqslant \frac{8M}{\sqrt{n}} \right)^3.$$

- $P_0\left(|Z| \leqslant \frac{8M}{\sqrt{n}}\right) = 2\int_0^{\frac{8M}{\sqrt{n}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \leqslant CM\frac{1}{\sqrt{n}}$.
- $\sum_{k=0}^{n-3} P_0(A_{n,k}) \leqslant (CM)^3 \frac{1}{\sqrt{n}}$.

命题3.3.10. 令$\mathrm{osc}(\delta) := \max_{t,s\leqslant 1,|t-s|\leqslant\delta}|B_t - B_s|$, 则

$$P\left(\limsup_{\delta\to 0}\frac{\mathrm{osc}(\delta)}{\sqrt{-\delta\log\delta}}\leqslant 6\right) = 1.$$

- $I_{n,m} = [\frac{m}{2^n}, \frac{m+1}{2^n}]$,

$$\Delta_{n,m} = \max_{t\in I_{n,m}}\left|B_t - B_{\frac{m}{2^n}}\right| \overset{d}{=} \frac{1}{\sqrt{2^n}}(M_1 \vee \hat{M}_1).$$

- $P(\Delta_{n,m} > \frac{1}{\sqrt{2^n}}x) \leqslant e^{-\frac{x^2}{2}}, \forall x \geqslant 4.$

$$\begin{aligned} \star\leqslant\ & 2P(M_1\geqslant x) = 2P(|Z|\geqslant x) \\ = \ & 4\int_x^\infty \sqrt{2\pi}^{-1} e^{-\frac{y^2}{2}}dy \leqslant \int_x^\infty y e^{-\frac{y^2}{2}}dy = e^{-\frac{x^2}{2}}. \end{aligned}$$

- $P(\exists m\leqslant 2^n - 1 \text{ s.t. } \Delta_{n,m} > \frac{1}{\sqrt{2^n}}x) \leqslant 2^n e^{-\frac{x^2}{2}}.$

- $P(\exists m \leqslant 2^n - 1 \text{ s.t. } \Delta_{n,m} > \frac{1}{\sqrt{2^n}} x) \leqslant 2^n e^{-\frac{x^2}{2}}$.
- 取$x_n = \sqrt{2(1+\varepsilon) n \log 2}$, 即$e^{-\frac{x_n^2}{2}} = 2^{-(1+\varepsilon)n}$, 则

$$P(A_n \text{ i.o.}) = 0, \quad A_n = \{\exists m \leqslant 2^n - 1 \text{ s.t. } \Delta_{n,m} > x_n/\sqrt{2^n}\}.$$

- $P(\Omega_1 = 1)$, $\forall \omega \in \Omega_1$:

$\exists N = N(\omega)$ s.t. $\forall n \geqslant N$, $\forall m \leqslant 2^n - 1$,

$$\Delta_{n,m} \leqslant \frac{1}{\sqrt{2^n}} \sqrt{2(1+\varepsilon) n \log 2} =: \varepsilon_n.$$

- 当$\delta \to 0$ 时: $\frac{1}{2^{n+1}} \leqslant \delta \leqslant \frac{1}{2^n}$, 其中$n \geqslant N$. 若$|t-s| \leqslant \delta$, 则

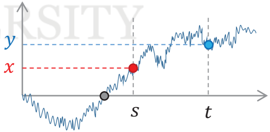
$$|B_t - B_s| \leqslant 3\varepsilon_n \leqslant 3\sqrt{2\delta}\sqrt{2(1+\varepsilon)\log \delta^{-1}} = 6\sqrt{(1+\varepsilon)\delta \log \delta^{-1}}.$$

4. 零点.

反正弦律(命题3.3.12). 令$L_t = \sup\{s \leqslant t : B_s = 0\}$, 则

$$P_0(L_t \leqslant s) = \frac{2}{\pi}\arcsin\sqrt{\frac{s}{t}}, \quad \forall\ 0 \leqslant s \leqslant t.$$

- L_t 为连续型, 密度为: $p(s) = \frac{1}{\pi\sqrt{t(t-s)}}$, $0 < s < t$.
- $P_0(L_t \leqslant s) = P\big(B_{s+u} \neq 0, \forall u \in [0, t-s]\big)$.
- $B_{s+u} = B_s + \hat{B}_u$, $\hat{B}_u := B_{s+u} - B_s$.
- 给定$B_s = x$ 的条件下, $\star$ 变为$\hat{\tau}_{-x} > t - s$.



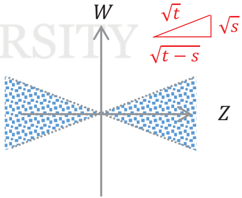
- $P_0(\tau_x \leqslant t-s) = P_0(|B_{t-s}| > |x|)$. 故,

$$P_0(\tau_x > t-s) = P_0(|B_{t-s}| \leqslant |x|) = P(\sqrt{t-s}|W| \leqslant |x|).$$

- $P_0(L_t \leqslant s) = P(\sqrt{t-s}|W| \leqslant \sqrt{s}|Z|) = \frac{2}{\pi} \arcsin \frac{\sqrt{s}}{\sqrt{t}}$.

- §3.1 习题3. $P_0(B_s > B_t > 0)$, $0 < s < t$.

- $P_0(L_t = 0) = 0$ vs $P_0(L_t > 0) = 1$.



推论3.3.10(零常返). $P_0(\sigma_0 = 0) = 1$, 其中

$$\sigma_0 = \inf\{t > 0 : B_t = 0\}.$$

- $P_0(\sigma_0 \leqslant t) = P_0(L_t > 0) = 1$, $\forall t > 0$. 故$P_0(\sigma_0 = 0) = 1$.
- 没有"首次返回"的时刻: $\sigma_0 = 0$ iff $\exists t_n \searrow 0$ 使得$B_{t_n} = 0$.
- 取$t_n = L_{\frac{1}{n}} \in (0, \frac{1}{n}]$.
- $\tau_{\frac{1}{n}} \searrow 0$: 在$\tau_{\frac{1}{n}}$ 与$\tau_{-\frac{1}{n}}$ 之间取t_n.

零点集: $\mathcal{Z} = \{t \geqslant 0 : B_t = 0\}$.

- $P_0(\exists t_n \in \mathcal{Z}\backslash\{0\}$ 使得$t_n \to 0) = 1$.
- $P_0(0 \in \mathcal{Z}') = 1$,

$$\mathcal{Z}' = \{t \geqslant 0 : \exists\, t_n \in \mathcal{Z}\backslash\{t\} \text{ s.t. } t_n \to t\}.$$

- 定义3.3.15. 假设$D \subseteq \mathbb{R}$, D 为闭集. 若$\forall t \in D$, 存在$t_n \in D\backslash\{t\}$, $\forall n$ 使得$t_n \to t$. 那么, 称D 为完全闭集.

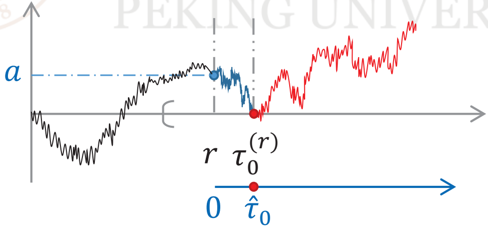
推论3.3.16. $P_0(\mathcal{Z}$ 是完全闭集$) = 1$.

- $\mathcal{Z} = \{0\} \cup C_- \cup C_+$,

$$C_- = \{t > 0 : \exists\, t_n \in \mathcal{Z} \text{ 使得} t_n \uparrow t\} \subseteq \mathcal{Z}',$$

$$C_+ = \{t > 0 : B_t = 0 \text{ 且} \exists \delta > 0 \text{ 使得} B_s \neq 0, \forall s \in (t-\delta, t)\}.$$

- $\forall t \in C_+$, $\exists r \in \mathbb{Q}_+$ 使得$t = \inf\{s \geqslant r : B_s = 0\} =: \tau_0^{(r)}$.
- 强马氏性: $\tilde{B}_t := B_{\tau+t}(-B_\tau)$ 是布朗运动.



$\dim(D)$: 用"尺寸"$=\varepsilon$的小区域覆盖, 需$N_\varepsilon \approx \varepsilon^{-d}$ 个.

- 例, Cantor集: $\dim(C) = \frac{\log 2}{\log 3}$.

  $\varepsilon = \frac{1}{3^n}$, $N_\varepsilon = 2^n = e^{n\log 2} = \varepsilon^{-\frac{\log 2}{\log 3}}$.

- $\varepsilon = \frac{1}{n}$. $I_i = [\frac{i}{n}, \frac{i+1}{n}]$, $i = 0, \dots, n-1$.

- 需要I_i: $\mathcal{Z} \cap I_i \neq \varnothing$.

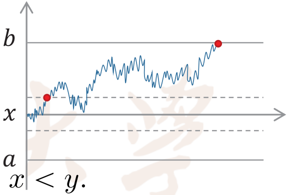
$$P_0(A_i) = \int p_{\frac{i}{n}}(x) P_x(\tau_0 < \frac{1}{n}) dx = \frac{4}{2\pi} \arcsin \frac{1}{\sqrt{i}} \approx \frac{1}{\sqrt{i}}.$$

- $\dim(\mathcal{Z}) = \frac{1}{2}$:

$$EN_\varepsilon \approx \sum_{i=1}^{n} \frac{1}{\sqrt{i}} \approx \sqrt{n} = n^{\frac{1}{2}}.$$

§3.4 位势理论

- 命题3.4.1. $P_x(\tau_b < \tau_a) = \frac{x-a}{b-a}$, $\forall a \leqslant x \leqslant b$.
- 证: 令$\varphi(x) = P_x(\tau_b < \tau_a)$.



- 强马: $\varphi(x) = \frac{1}{2}\big(\varphi(x+\delta) + \varphi(x-\delta)\big)$.
- 单调: $\varphi(x) = P_x(\tau_y < \tau_a)\varphi(y) \leqslant \varphi(y)$, $x < y$.
- 推论3.4.2(Wald 引理). $E_x B_\tau = x$, $\tau = \tau_a \wedge \tau_b$.
- 证: $E_x B_\tau = b P_x(B_\tau = b) + a P_x(B_\tau = a)$

$$= b \cdot \frac{x-a}{b-a} + a \cdot \frac{b-x}{b-a} = x.$$

- 引理3.4.3. $E_x\tau < \infty, \forall x \in [a,b]$.

$$P_x(\tau \geqslant n) \leqslant P_x(B_1,\dots,B_n \in [a,b]) \leqslant \delta^n,$$

$$\text{其中,}\ \delta \leqslant \max_{a\leqslant y\leqslant b} P(y+Z \in [a,b]).$$

- 命题3.4.4(Wald第二引理). $E_x(B_\tau - x)^2 = E_x\tau$.
- 计算: $E_x\tau = (b-x)^2 \cdot \frac{x-a}{b-a} + (x-a)^2 \cdot \frac{b-x}{b-a}$
  $= (x-a)(b-x)\frac{b-a}{b-a} = (x-a)(b-x)$.
- 特别地, $E_0\tau_{\pm\delta} = \delta^2$.

不变原理的证明.

- $x=0$, $\sigma := \inf\{t \geqslant 0 : |B_t| = \frac{1}{\sqrt{N}}\}$, $E\sigma = \frac{1}{N}$. $\sigma_1, \sigma_2, \cdots$
- 随机游动的尺度变换: $S^{(N)}_{n/N} := B_{T_n}$, $T_n = \sigma_1 + \cdots + \sigma_n$.
- $\frac{n}{N} \leqslant t \leqslant \frac{n+1}{N}$, $S^{(N)}_t$: $t^* = T_n$ 和T_{n+1} 的B_{t^*} 的线性插值.
- 结论: $\max_{0\leqslant t\leqslant 1} |t^* - t| \xrightarrow{P} 0$, 故$\max_{0\leqslant t\leqslant 1} |S^{(N)}_t - B_t| \xrightarrow{P} 0$.
- 证: $\star \leqslant \frac{1}{N} + \max_{0\leqslant n\leqslant N} |T_n - \frac{n}{N}|$.

$$\star = \max_{0\leqslant n\leqslant N} \frac{n}{N}\left|\frac{NT_n}{n} - 1\right| \overset{d}{\leqslant} \varepsilon \max_{n\geqslant 1}\left|\frac{W_n}{n} - 1\right| + \max_{n\geqslant \varepsilon N}\left|\frac{W_n}{n} - 1\right|$$

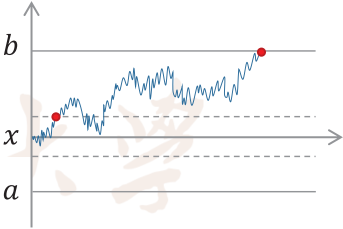
- 一般的随机游动: $E\xi = 0$, $E\xi^2 = 1$.

  则, 存在停时τ 使得$E\tau < \infty$, $B_\tau \overset{d}{=} \xi$.

命题3.4.5.I. $\varphi(x) = E_x f(B_\tau)$ 满足:

$$\varphi''(x) = 0, \qquad \varphi(a) = f(a),\ \varphi(b) = f(b).$$

- 例: $f(y) = 1_{\{y=b\}}$, $\varphi(x) = P_x(\tau_b < \tau_a)$.
- $\varphi(x) = \frac{1}{2}\big(\varphi(x+\delta) + \varphi(x-\delta)\big)$.



命题3.4.5.II. $\psi(x) := E_x \int_0^\tau g(B_t)dt$ 满足:

$$\psi''(x) = -2g(x), \qquad \psi(a) = \psi(b) = 0.$$

- 例: $g \equiv 1$, $\psi(x) = E_x\tau$.
- $\psi(x) = E_x \int_0^{\sigma_\delta} g(B_t)dt + \frac{1}{2}\big(\psi(x+\delta) + \psi(x-\delta)\big)$.
- $\psi(x+\delta) + \psi(x-\delta) - 2\psi(x) \approx -2g(x) \times \delta^2$.

$D \subseteq \mathbb{R}^d$, $\tau = \tau_{\partial D}$.

- 命题3.4.6. $\varphi(x) = E_x f(B_\tau)$ 是如下Dirichlet 问题的解:

$$\begin{cases} \Delta\varphi(x) = 0, & \forall\ x \in D, \\ \lim\limits_{y \in D, y \to x} \varphi(y) = f(x), & \forall\ x \in \partial D. \end{cases}$$

- 命题3.4.7. $\psi(x) := E_x \int_0^\tau g(B_t)dt$ 是如下Poisson 问题的解:

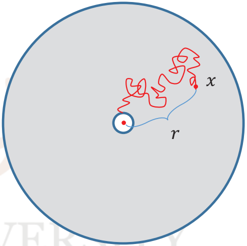
$$\begin{cases} \Delta\psi(x) = -2g(x), & \forall\ x \in D, \\ \lim\limits_{y \in D, y \to x} \psi(y) = 0, & \forall\ x \in \partial D. \end{cases}$$

例3.4.9. $\mathbb{R}^d$, $\tau_a = \inf\{t \geqslant 0 : \|B_t\| = a\}$. 求$P_x(\tau_\varepsilon < \tau_R)$.

- $D = \{x : \varepsilon \leqslant \|x\| \leqslant R\}$. $\tau = \tau_{\partial D}$.
- $\varphi(x) = E_x f(B_\tau)$,

$$f(y) = \begin{cases} 1, & 若\|y\| = \varepsilon; \\ 0, & 若\|y\| = R. \end{cases}$$

- $\Delta\varphi(x) = 0$, $x \in D$; $\quad \varphi|_{\partial D} = f$.
- 由各项同性: 令$\varphi(x) = \Phi(s)$, 其中 $s = r^2 = \|x\|^2 = \sum_{i=1}^d x_i^2$.



例3.4.9(续). $\mathbb{R}^d$, $\tau_a = \inf\{t \geqslant 0 : \|B_t\| = a\}$. 求$P_x(\tau_\varepsilon < \tau_R)$.

- $s = r^2$, $\varphi(x) = \Phi(s)$. $4r^2\Phi''(s) + 2d\Phi'(s) = 0$:

$$\frac{\partial\varphi(x)}{\partial x_i} = \Phi'(s)\frac{\partial s}{\partial x_i} = \Phi'(s)\cdot 2x_i,$$
$$\frac{\partial^2\varphi(x)}{\partial^2 x_i} = \Phi''(s)(2x_i)^2 + \Phi'(s)\cdot 2.$$

- 令$\Phi' = \Psi$: $\quad 2s\Psi'(s) + d\Psi(s) = 0.$

$$(\ln\Psi(s))' = -\frac{d}{2}\cdot\frac{1}{s} = (-\frac{d}{2}\ln s)' \Rightarrow \Psi(s) = C\cdot s^{-d/2}.$$

- $\Phi(\varepsilon^2) = 1$, $\Phi(R^2) = 0$,

$$P_x(\tau_\varepsilon < \tau_R) = \Phi(s) = C_1\int\frac{1}{s^{d/2}}ds + C_2 = \frac{f(R) - f(r)}{f(R) - f(\varepsilon)}.$$

- $d = 1$, $f(r) = \sqrt{s} = r$; $d = 2$, $f(r) = \ln s = 2\ln r$;

  $d = 3$, $f(r) = s^{-d/2+1} = \frac{1}{r^{d-2}}$.

- $d=1$. $f(r)=r$, $P_x(\tau_\varepsilon<\tau_R)=\frac{R-r}{R-\varepsilon}$.
- $P_x(\tau_0<\tau_R)=\lim\limits_{\varepsilon\to 0}P_x(\tau_\varepsilon<\tau_R)=\frac{R-r}{R}$.
- 再令$R\to\infty$: $P_x(\tau_0<\infty)=\lim\limits_{R\to 0}P_x(\tau_0<\tau_R)=1$, 点常返.
- $d=2$. $\Phi(r^2)=C_1\ln s+C_2=\frac{\ln R-\ln r}{\ln R-\ln\varepsilon}$.
- 令$\varepsilon\to 0$: $P_x(\tau_0<\tau_R)=\lim\limits_{\varepsilon\to 0}P_x(\tau_\varepsilon<\tau_R)=0$.
  再令$R\to\infty$: $P_x(\tau_0<\infty)=\lim\limits_{R\to\infty}P_x(\tau_0<\tau_R)=0$.
- 令$R\to\infty$: $P_x(\tau_\varepsilon<\infty)=\lim\limits_{R\to 0}P_x(\tau_\varepsilon<\tau_R)=1$, 集合常返.
- $d\geqslant 3$. $\Phi(r^2)=C_1\frac{1}{s^{d/2-1}}+C_2=\frac{\frac{1}{r^{d-2}}-\frac{1}{R^{d-2}}}{\frac{1}{\varepsilon^{d-2}}-\frac{1}{R^{d-2}}}$.
- 令$R\to\infty$: $P_x(\tau_\varepsilon<\infty)=(\frac{\varepsilon}{r})^{d-2}<1$, 非常返.

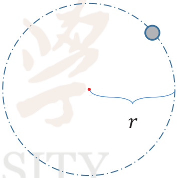
- $d \geqslant 3$, $P_x(\tau_\varepsilon < \infty) = (\frac{\varepsilon}{r})^{d-2}$:

$$P_x(\tau_e < \tau_R) = \frac{\frac{1}{r^{d-2}} - \frac{1}{R^{d-2}}}{\frac{1}{\varepsilon^{d-2}} - \frac{1}{R^{d-2}}} \xrightarrow{R\to\infty} (\frac{\varepsilon}{r})^{d-2}.$$

- 用边长为$\varepsilon = \frac{1}{N}$ 的小方块覆盖:

  $d(x,o) \approx r = k\varepsilon$ 的方块共$\approx Ck^{d-1}$ 个,

  平均需要其中的$\approx Ck^{d-1}(\frac{\varepsilon}{r})^{d-2} = Ck$ 个,



- $\sum_{k=1}^{\alpha N} Ck \approx C\alpha^2 N^2$. 即, $\dim(\mathcal{R}_d) = 2$.
- $\dim(\mathcal{R}_d) = \dim\big(\mathcal{G}_d\big) = 2, \forall d \geqslant 2$. $\dim(\mathcal{R}_1) = 1$, $\dim(\mathcal{G}_1) = \frac{3}{2}$.

例3.4.10. $D=[0,1]$, $\tau=\tau_{\partial D}$. 求

$$\psi(x;y,z):=E_x\int_0^\tau 1_{\{B_t\in(y,z)\}}dt.$$

- 固定y,z. $\psi(x):=\psi(x;y,z)$ 满足: $\psi''(x)=-2$, 若$y\leqslant x\leqslant z$; $\psi''(x)=0$, 其他. $\psi(0)=\psi(1)=0$.
- 令$h(x)=-x^2+ax+b$,
- $\psi'(y)=-2y+a=\frac{h(y)}{y}$,
  $\psi'(z)=-2z+a=-\frac{h(z)}{1-z}$

- $a=2z-(z^2-y^2)$, $b=-y^2$. 解得ϕ.
- $\psi(x)=\int_y^z\rho_x(w)dw$, $\rho_x(w)=2(x\wedge w)(1-x\vee w)$.

例3.4.10(续) $\psi(x;y,z) := E_x \int_0^\tau 1_{\{B_t\in(y,z)\}}dt.$

- 占有时(occupation time)、格林函数. $D \subseteq \mathbb{R}^d$, $\tau = \tau_{\partial D}$,

$$\mu_x^D(A) = E_x \int_0^\tau 1_{\{B_t\in A\}}dt.$$

- $\mu_x^D(A) = E_x \int_0^\infty 1_{\{B_t\in A,\tau>t\}}dt = \int_0^\infty P_x(B_t \in A, \tau > t)dt.$
- $P_x(B_t \in A, \tau > t) = \int_A p_t^D(x,y)\, dy.$
- $\mu_x^D(A) = \int_A G^D(x,y)dy$, $G^D(x,y) = \int_0^\infty p_t^D(x,y)dt.$
- $G^{[0,1]}(x,y) = G^{[0,1]}(y,x) = 2x(1-y)$, $0 \leqslant x \leqslant y \leqslant 1.$

例: $d=1$. 计算$G^D(x,y)$, $D=\mathbb{R}_+$.

- 反射原理: $P_x(\tau\leqslant t, B_t\in A)=P_x(\tau\leqslant t, B_t\in -A)$.
- $P_x(B_t\in A,\tau>t)=P_x(B_t\in A)-P_x(B_t\in -A)$
  $$=\int_A p_t(x,y)dy-\int_A p_t(x,-y)dy.$$
- $p_t^D(x,y)=\frac{1}{\sqrt{2\pi t}}\left(e^{-\frac{(y-x)^2}{2t}}-e^{-\frac{(-y-x)^2}{2t}}\right)$.
- $G^D(x,y)=\int_0^\infty p_t^D(x,y)dt=(y+x)-|y-x|=2(x\wedge y)$.

$$\varphi(a,b):=\int_0^\infty\frac{1}{\sqrt{2\pi t}}(e^{-\frac{a}{2t}}-e^{-\frac{b}{2t}})dt=\int_a^b\int_0^\infty\frac{1}{\sqrt{2\pi t}}\frac{1}{2t}e^{-\frac{z}{2t}}dtdz$$
$$=\int_a^b\frac{-2}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{z}{2\sqrt{t}^2}}d\frac{1}{\sqrt{t}}dz$$
$$=\int_a^b\frac{1}{2\sqrt{z}}\frac{2\sqrt{z}}{\sqrt{2\pi}}\int_0^\infty e^{-\frac{zs^2}{2}}dsdz=\int_a^b\frac{1}{2\sqrt{z}}dz=\sqrt{b}-\sqrt{a}.$$

例: $d=2$. $D=\mathbb{H}=\mathbb{R}\times\mathbb{R}_+$.

- 反射原理: $p_t^D(x,y)=p_t(x,y)-p_t(x,\tilde{y})$, $\tilde{y}=(y_1,-y_2)$.
- $p_t^D(x,y)=\frac{1}{2\pi t}\left(e^{-\frac{\|y-x\|^2}{2t}}-e^{-\frac{\|\tilde{y}-x\|^2}{2t}}\right)$
- $\star=e^{-\frac{(y_1-x_1)^2}{2t}}\left(e^{-\frac{(y_2-x_2)^2}{2t}}-e^{-\frac{(-y_2-x_2)^2}{2t}}\right)$,

  $\int_a^b \frac{1}{2t}e^{-\frac{z}{2t}}dz$, $a=(y_2-x_2)^2$, $b=(-y_2-x_2)^2$.
- $G^D(x,y)=\frac{1}{\pi}\ln\frac{\|x-\tilde{y}\|}{\|x-y\|}$.

$$G^D(x,y)=\int_a^b\int_0^\infty \frac{1}{2\pi t}\cdot\frac{1}{2t}e^{-\frac{z+(y_1-x_1)^2}{2t}}dtdz$$

$$=\frac{1}{4\pi}\int_a^b\int_0^\infty e^{-\frac{z+(y_1-x_1)^2}{2}\cdot s}dsdz$$

$$=\frac{1}{4\pi}\int_a^b\frac{2}{z+(y_1-x_1)^2}dz=\frac{1}{2\pi}\ln(z+(y_1-x_1)^2)\Big|_a^b.$$

- $G^D(x,y) = \frac{1}{2\pi}\ln(z+(y_1-x_1)^2)\mid_a^b$, $a=(y_2-x_2)^2$, $b=(-y_2-x_2)^2$.

$$\pi G^D(x,y) = \ln\frac{\|x-\tilde{y}\|}{\|x-y\|}.$$

- $\pi G^D(x,y) \overset{y\to\partial\mathbb{H}}{\longrightarrow} 0$.
- $\pi G^D(x,y) = \ln\frac{1}{\|x-y\|} + \ln\|x-\tilde{y}\| \overset{y\to x}{\longrightarrow} \infty$.
- $\Delta\int G^D(x,y)g(y)dy = -2g(x)$.
- 黎曼映照定理: $\exists$ 共形映照$\Phi: D\to\mathbb{H}$, $x\mapsto\hat{x}$. 则,

$$G^D(x,y) = G^{\mathbb{H}}(\Phi(x),\Phi(y)).$$