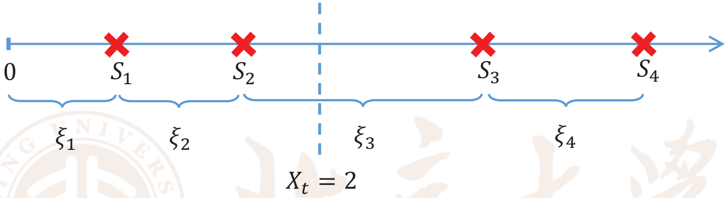
第二章、跳过程

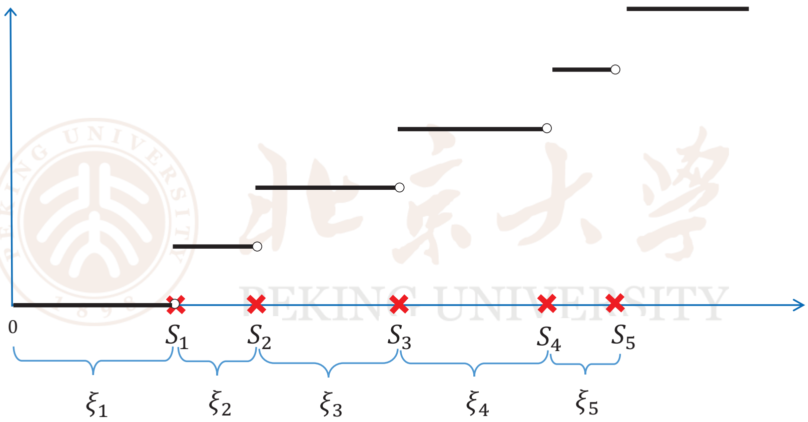
§2.1 泊松过程(Poisson Process)



- $\xi_1, \xi_2, \ldots$ i.i.d. $\sim \text{Exp}(\lambda)$. $P(\xi > t) = e^{-\lambda t}$.
- $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$, 其中$S_0 := 0$.
- $X_t := |\{n \geqslant 1 : S_n \leqslant t\}| = \sup\{n \geqslant 0 : S_n \leqslant t\}$.
- $X_t = k$ iff $S_k \leqslant t < S_{k+1}$. ($X_t < k$ iff $S_k > t$.)
- 泊松过程: $\{X_t, t \geqslant 0\}$, λ: 参数、强度.

  泊松点过程/泊松流: $\Xi = \{S_1, S_2, \ldots\}$, 记为$\sim \mathbf{PP}(\lambda)$.

轨道图: $X_t := |\Xi \cap [0,t]|$, $\Xi = \{S_1, S_2, \ldots\}$.



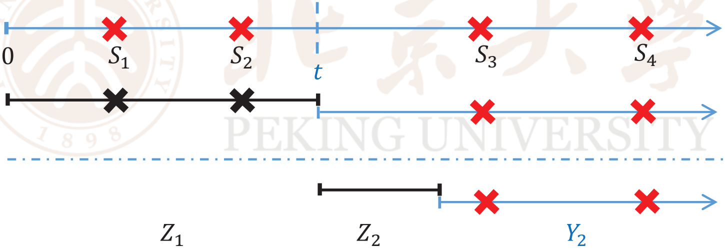
- $\forall t \geqslant 0$, $P(t \in \Xi) = P(\bigcup_n \{S_n = t\}) \leqslant \sum_{n=1}^{\infty} P(S_n = t) = 0$.

命题 (定理2.1.2, 定理2.1.3, 命题2.1.9)

令$Z = [0,t] \cap \{S_1, S_2, \ldots\}$,

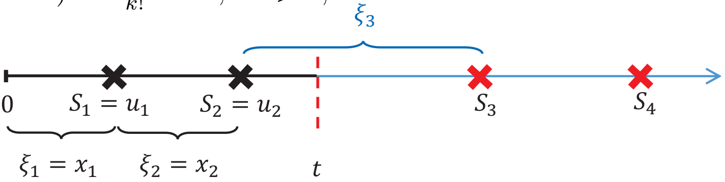
$$Y = \theta_t \Xi := [0,\infty) \cap \{S_1 - t, S_2 - t, \ldots\}.$$

则$X_t = |Z| \sim \mathcal{P}(\lambda t)$, "$\cdots$"; $Y \sim \mathbf{PP}(\lambda)$; 且$Y$, Z 相互独立.



- 独立平稳增量过程(推论2.1.5, 定义2.1.16), 马氏过程(推论2.1.7).

验证$P(|Z|=k)=\frac{(\lambda t)^k}{k!}e^{-\lambda t}, \forall k\geqslant 0$; "...".



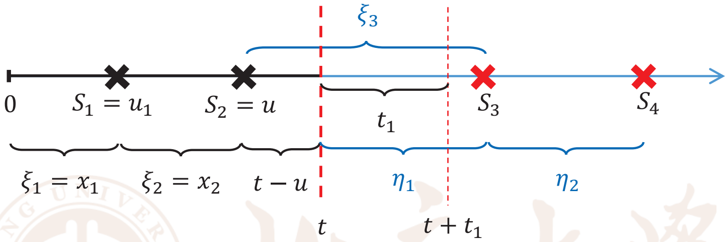
- 假设$k\geqslant 1$, $0<u_1<\ldots<u_k<t$,

  $P(S_1\approx u_1,\ldots,S_k\approx u_k)=p_{S_1,\ldots,S_k}(u_1,\ldots,u_k)du_1\cdots du_k$.
- $P(|Z|=k;A)=P(A;\xi_{k+1}>t-S_k)=e^{-\lambda t}\cdot e^{\lambda u_k}P(A)$,

  $p_{S_1,\ldots,S_k}(u_1,\ldots,u_k)=\lambda^k e^{-\lambda(x_1+\ldots+x_k)}=\lambda^k e^{-\lambda u_k}$.
- $P(|Z|=k;A)=\lambda^k e^{-\lambda t}1_{\{0<u_1<\ldots<u_k<t\}}du_1\cdots du_k$, $\quad(\to \frac{t^k}{k!})$.
- $|Z|\sim\mathcal{P}(\lambda t)$, $P(A||Z|=k)=\frac{k!}{t^k}1_{\{0<u_1<\ldots<u_k<t\}}du_1\cdots du_k$.
- $Z\overset{d}{=}\{U_1,\ldots,U_W\}$, 其中$U_1,U_2,\ldots$ i.i.d., $\sim U(0,t)$,

  $W\sim\mathcal{P}(\lambda t)$, 且所有随机变量相互独立.

验证$Y \sim \mathbf{PP}(\lambda)$, 且$Y$, Z 相互独立.



- 令$B = \{\eta_1 > t_1, \eta_2 > t_2, \ldots, \eta_m > t_m\}$. 往验证

$$P(\underline{|Z| = k, A}; \boxed{B}) = P(\underline{|Z| = k, A}) \times \boxed{e^{-\lambda(t_1+\ldots+t_m)}}.$$

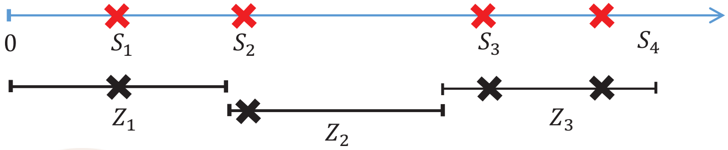
- 令$\hat{B} = \{S_{k+1} > t + t_1; \xi_{k+2} > t_2, \ldots, \xi_{k+m} > t_m\}$. 则

$$\{|Z| = k; B\} = \{|Z| = k; \hat{B}\}.$$

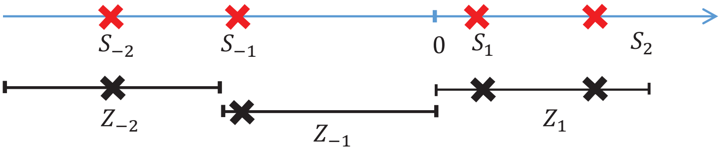
- 左边$= P(\underline{A, S_{k+1} > t}; S_{k+1} > t + t_1, \tilde{B})$

$$= P(|X_{t+t_1}| = k, A; \tilde{B}) = P(A; |Z| = k) \times e^{-\lambda t_1} \cdot P(\tilde{B}).$$

构造泊松流(命题2.1.10):

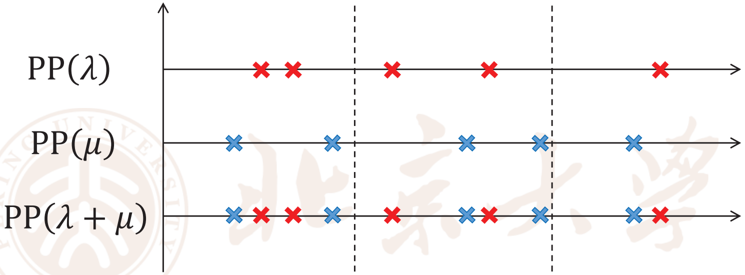


- $Z_1 = \Xi \cap [0,1], \quad \theta = \theta_1 : \Xi \to Y.$
- $Z_2 = \theta(\Xi) \cap [0,1], \quad Z_3 = \theta^{(2)}(\Xi) \cap [0,1], \quad \ldots$
- $Z_1, Z_2, \ldots$ i.i.d., $\quad \Xi = Z_1 \vee Z_2 \vee \ldots.$
- $Z_n, n \in \mathbb{Z}$, i.i.d., $\quad \Xi = \ldots \vee Z_{-2} \vee Z_{-1} \vee Z_1 \vee Z_2 \vee \ldots.$



泊松流的合并与细分:

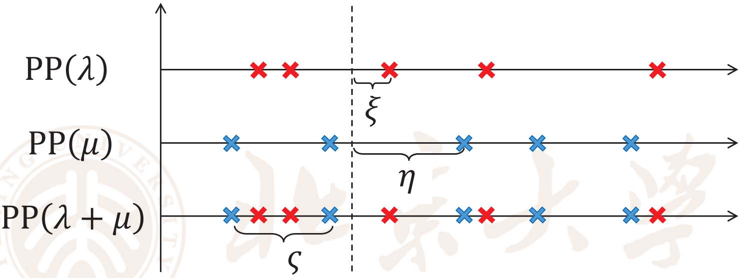
$\Xi \sim \mathbf{PP}(\lambda)$ 与 $\Theta \sim \mathbf{PP}(\mu)$ 相互独立 vs $\Xi \cup \Theta \sim \mathbf{PF}(\lambda+\mu)$.



- $\Xi = \tilde{Z}_1 \vee \tilde{Z}_2 \vee \ldots,\quad \Theta = \hat{Z}_1 \vee \hat{Z}_2 \vee \ldots,$
  $Z_n \overset{d}{=} \{\tilde{U}_1, \ldots, \tilde{U}_{\tilde{W}}\},\quad \hat{Z}_n \overset{d}{=} \{\hat{U}_1, \ldots, \hat{U}_{\hat{W}}\},$
- $Z_n = \tilde{Z}_n \cup \hat{Z}_n,\quad \tilde{Z}_1 \overset{d}{=} \{U_1, \ldots, U_W\},\ W = \tilde{W} + \hat{W}.$
- $\Xi \cup \Theta = Z_1 \vee Z_2 \vee \ldots \sim \mathbf{PP}(\lambda+\mu).$

泊松流的合并与细分:

$\Xi \sim \mathbf{PP}(\lambda)$ 与$\Theta \sim \mathbf{PP}(\mu)$ 相互独立 vs $\Xi \cup \Theta \sim \mathbf{PF}(\lambda+\mu)$.



- $\zeta = \xi \wedge \eta$:

$$P(\zeta > t) = e^{-\lambda t} \times e^{-\mu t}, \quad P(\xi < \eta) = \frac{\lambda}{\lambda+\mu} = p.$$

- $\zeta \sim \mathrm{Exp}(\lambda+\mu)$, $V \sim \mathcal{G}(p)$.

$$\zeta_1 + \cdots + \zeta_V \sim \mathrm{Exp}(\lambda).$$

泊松点过程Ξ:

- $\mathbb{R}^d$ 上: (i) $|\Xi \cap D| \sim \mathcal{P}(\lambda|D|)$,
  (ii) 若$D_1, \ldots, D_n$ 互不交, 则$|\Xi \cap D_i|$, $i = 1, \ldots, n$ 独立.
- $(\mathbb{S}, \mu(\cdot))$ 上: (i) $|\Xi \cap D| \sim \mathcal{P}(\mu(D))$, (ii).
- 构造:
  (a) 将$\mathbb{S}$ 划分为$D_1, D_2, \cdots$ 使得$0 < \mu(D_n) < \infty$;
  (b) 取$U_{n1}, U_{n2}, \ldots \overset{\text{i.i.d.}}{\sim} \frac{1}{\mu(D_n)}\mu(\cdot)$, $W_n \sim \mathcal{P}(\mu(D_n))$;
  (c) $\Xi = \bigcup_n \{U_{n1}, \ldots, U_{nW_n}\}$.
- 例: $\mathbb{R}_1$ 上, $\mu\big((a,b)\big) = \int_a^b f(x)dx$. 则

$$P(\Xi \cap [a,b] = \varnothing) = e^{-\int_a^b f(x)dx}.$$

§2.2 跳过程的构造及其转移概率

1. 定义

- $\forall i \in S$,

  一组小闹钟q_{ij}, $j \neq i$; 或

  大闹钟q_i 与色子$\hat{p}_{ij}$, $j \neq i$:

$$q_i = \sum_{j \neq i} q_{ij} < \infty, \quad \hat{p}_{ij} = \frac{q_{ij}}{q_i}.$$

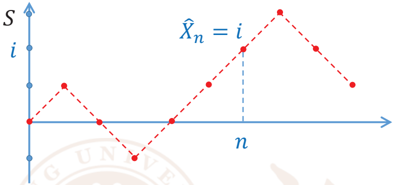
- 特殊情况: 吸收态, $q_i = 0$, $\hat{p}_{ii} := 1$.

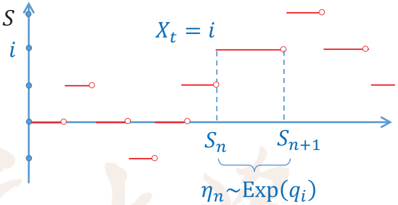
  非吸收态: $q_i > 0$, $\hat{p}_{ii} := 0$.

- 速率矩阵(定义2.2.1):

$$\mathbf{Q} = (q_{ij})_{S \times S}, \quad q_{ii} := -q_i.$$

- 色子$\hat{\mathbf{P}} = (\hat{p}_{ij})_{S\times S}$: 嵌入链$\{\hat{X}_n\}$;　　闹钟: 时间变换.





- $\xi_0, \xi_1, \ldots$ i.i.d. $\sim \mathrm{Exp}(1)$, 与$\{Y_n\}$ 独立.

$$\eta_0 = \frac{\xi_0}{q_{Y_0}},\ \eta_1 = \frac{\xi_1}{q_{Y_1}}, \ldots.$$

- 在$\{\hat{X}_n\} = \vec{i}$ 的条件下,

$$\eta_0, \eta_1, \ldots \text{ 相互独立, } \eta_n \sim \mathrm{Exp}(q_{i_n}).$$

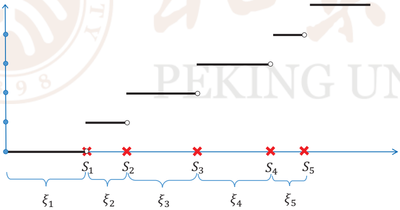
- $X|_{[S_0,S_1)} := \hat{X}_0,\ \ X|_{[S_1,S_2)} := \hat{X}_1,\ \ldots \quad S_n = \eta_0 + \ldots + \eta_{n-1}.$
- 最小过程: $X|_{[\tau_\infty,\infty)} = \partial, \quad \tau_\infty :=\uparrow \lim_n S_n$ 为爆炸时.

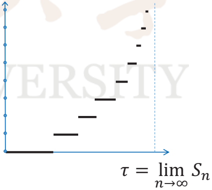
2. 爆炸与非爆炸.

- 命题2.2.3. $\zeta_1, \zeta_2, \cdots$ 独立, $\zeta_n \sim \operatorname{Exp}(\lambda_n)$, $\tau = \sum_n \zeta_n$. 则

$$\sum_i \frac{1}{\lambda_i} = \infty \quad \text{iff} \quad P(\tau = \infty) = 1,$$

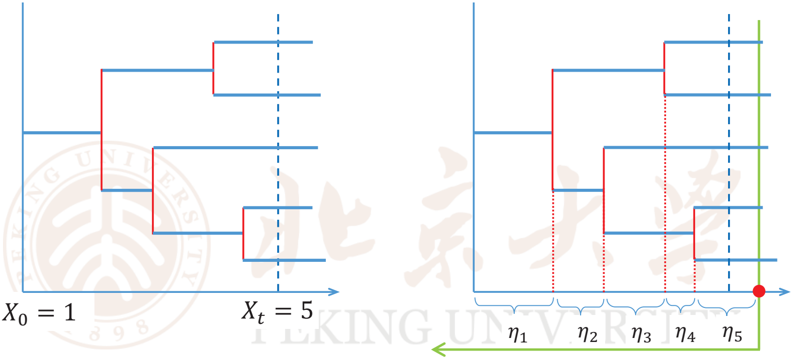
$$\sum_i \frac{1}{\lambda_i} < \infty \quad \text{iff} \quad P(\tau < \infty) = 1.$$





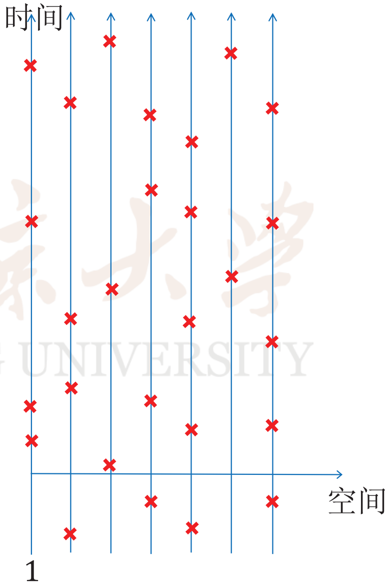
- 推论2.2.7. $q_i \leqslant \lambda \cdot |i|$, $\forall i \neq o$, 则非爆炸.

例2.2.4 & 2.2.6. Yule过程/纯生过程. $\lambda_i = i\lambda,\ i \geqslant 1$. 非爆炸.

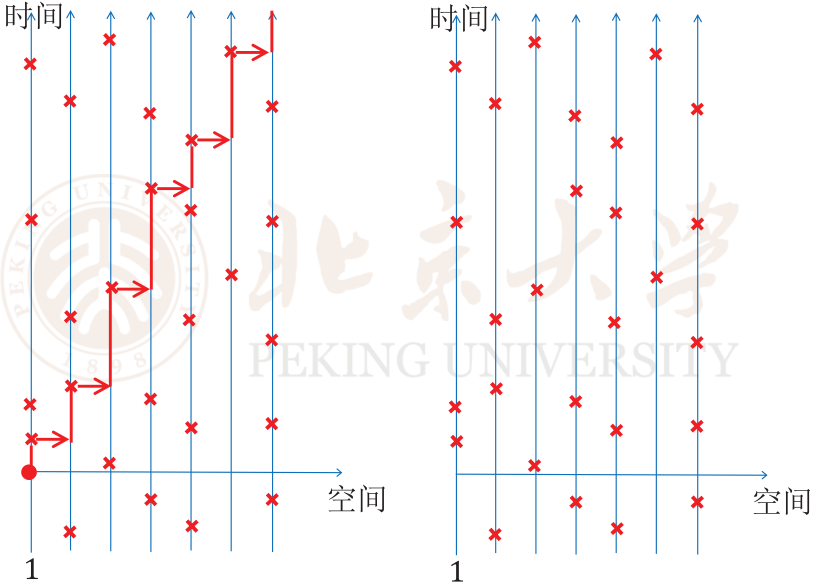


- $X_t \leqslant i$ iff $\eta_1 + \cdots + \eta_i > t$.
- $\eta_1 + \cdots + \eta_i > t$ iff 存活.
- $P(X_t \geqslant i+1) = (1 - e^{-\lambda t})^i,\ \forall i \geqslant 1$.

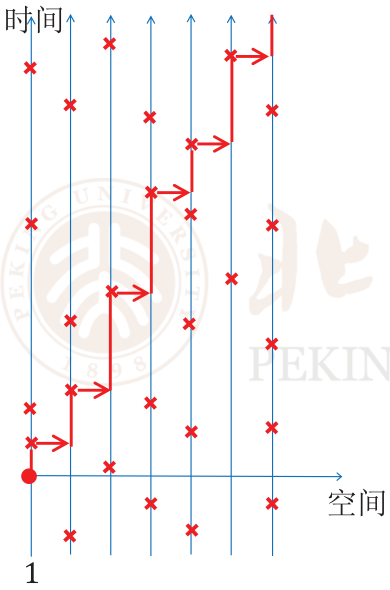
图表示与对偶:



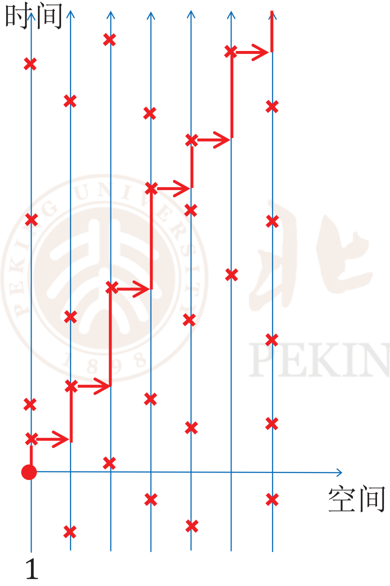
图表示与对偶:

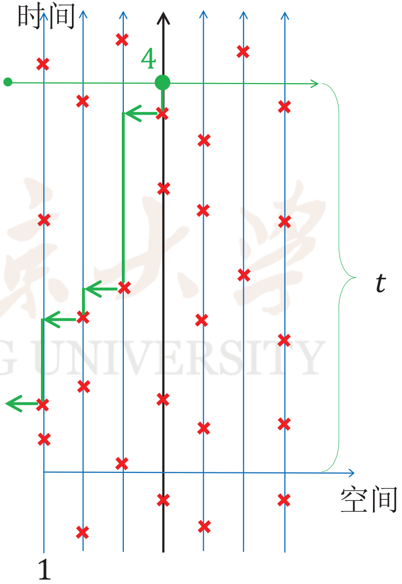


图表示与对偶:



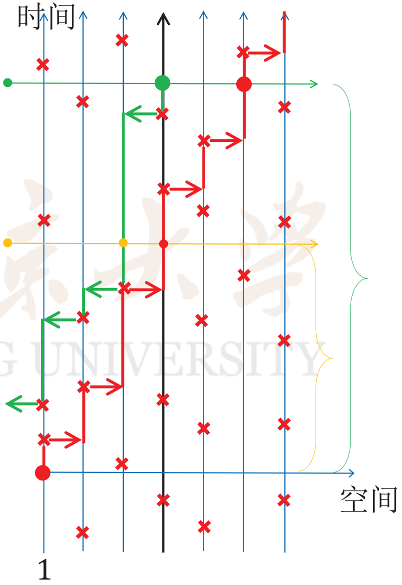
图表示与对偶:





图表示与对偶:

- $X_t \geqslant i+1$ iff $Y_t^{(i)}=0.$
- 概率$=(1-e^{-\lambda t})^i.$



3. 转移概率与转移速率.

- 转移概率 $\mathbf{P}(t) = (p_{ij}(t))_{S\times S}$:

$$p_{ij}(t) = P_i(X_t = j) = \sum_{n=0}^{\infty} P_i(X_t = j, \underline{N_t = n}).$$

- $P_i(\hat{X}_1 = i_1, \dots, \hat{X}_n = i_n; \underline{S_n \leqslant t < S_{n+1}})$
- $\star\star = p_{i_0 i_1} \cdots p_{i_{n-1}, i_n} P(** | **)$,
- $\star\star = \int_{\Delta_n} q_{i_0} e^{-q_{i_0} x_0} \cdots q_{i_n} e^{-q_{i_n} x_n} dx_0 \cdots dx_n$,

  $\Delta_n := \{\vec{x} : x_i \geqslant 0, \forall i; x_0 + \cdots + x_{n-1} \leqslant t < x_0 + \cdots + x_n\}$.
- 非爆炸: $\sum_j p_{ij}(t) = 1, \forall i, t$.

$$\sum_{j\in S} p_{ij}(t) = P_i(X_t \in S) = P_i(\tau_\infty > t) \leqslant 1.$$

- 定理2.2.11.

  $P_{i_0}(X_{t_1}=i_1,\cdots,X_{t_n}=i_n)=\prod_{r=0}^{n-1} p_{i_r i_{r+1}}(t_{r+1}-t_r).$
- 推论2.2.12. Chapman-Kolmogorov 等式:

$$\mathbf{P}(t+s)=\mathbf{P}(t)\mathbf{P}(s), \quad \text{即 } p_{ij}(t+s)=\sum_k p_{ik}(t)p_{kj}(s).$$

- 定理2.2.13. $\mathbf{P}'(0)=\mathbf{Q}$, 即$p'_{ij}(0)=q_{ij}, \forall i,j.$

$$\frac{1}{t}P_i(N_t=1, X_t=j)\to q_{ij};$$

$$\frac{1}{t}P_i(N_t=1)\to q_i, \quad \frac{1}{t}P_i(N_t\geqslant 2)\to 0.$$

- 命题2.2.15. Kolmogorov 前进、后退方程:

$$\mathbf{P}'(t)=\mathbf{P}(t)\mathbf{Q}=\mathbf{Q}\mathbf{P}(t).$$

- 后退方程$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$ 的应用. f 是S 上的函数.

$$f_t := \mathbf{P}(t)f, \quad f_t(i) := \sum_k p_{ik}(t)f(k) = E_i f(X_t),$$

$$f_t' = (\mathbf{P}(t)f)' = \mathbf{Q}\mathbf{P}(t)f = \mathbf{Q}f_t$$

- 前进方程$\mathbf{P}'(t) = \mathbf{P}(t)\mathbf{Q}$ 的应用. μ 是S 上的分布.

$$\mu_t = \mu\mathbf{P}(t), \quad \mu_t(i) := P_\mu(X_t = i),$$

$$\mu_t' = (\mu\mathbf{P}(t)) = \mu\mathbf{P}'(t) = \mu\mathbf{P}(t)\mathbf{Q} = \mu_t\mathbf{Q}.$$

补充知识:

- 定义2.2.19. (强连续)马氏半群$\{\mathbf{P}(t) : t \geqslant 0\}$:

  (1) $\mathbf{P}(0) = \mathbf{I}$, (2) $\mathbf{P}(t+s) = \mathbf{P}(t)\mathbf{P}(s)$,

  (3) $\lim_{t\to 0}\mathbf{P}(t) = \mathbf{P}(0)$, $p_{ij}(t) \to p_{ij}(0)$, $\forall i, j$.

- 命题2.2.22. $\mathbf{Q} = \mathbf{P}'(0)$ 存在且满足:

  $0 \leqslant q_{ij} < \infty$; $0 \leqslant q_i := -q_{ii} \leqslant \infty$; $\sum_{j\neq i} q_{ij} \leqslant q_i$.

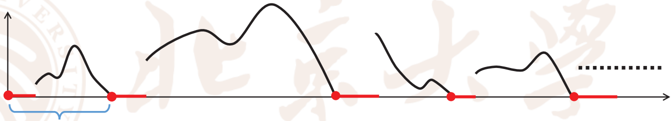
- 命题2.2.24. $\mathbf{Q}$ 保守, 则后退方程$\mathbf{P}'(t) = \mathbf{Q}\mathbf{P}(t)$ 成立.
- 连续时间马氏链$\{X_t\}$: 以$\{\mathbf{P}(t), t \geqslant 0\}$ 为转移矩阵.
- 生成元: 定义域为$\mathcal{D} = \mathcal{D}(\mathcal{L}) \subseteq 2^S$,

$$\mathcal{L} : \mathcal{D} \to 2^S, \quad f \mapsto \mathcal{L}(f) = \mathbf{Q}f,$$

$$\frac{d}{dt} f_t = \mathcal{L}(f_t), \quad f_t(i) := E_i f(X_t).$$

§2.3 首达时、吸收概率

- 不可约: 可达、互通. (定义2.3.1, 命题2.3.2)
- 强马氏性(引理2.3.3 & 2.4.2). 令$\tau = \tau_i$ 或σ_i. 在$\tau < \infty$ 的条件下, $\{Y_t := X_{\tau+t}\}$ 为从i 出发的跳过程, 且与$X_{[0,\tau]}$ 独立.



- $x_i = P_i(\tau_o < \infty)$: $x_i = \sum_j p_{ij}x_j$, $i \neq o$; $x_o = 1$.
- $y_i = E_i\tau_o$: $y_i = \frac{1}{q_i} + \sum_j p_{ij}y_j$, $i \neq 0$; $y_o = 0$.
- $z_i = E_i \int_0^{\tau_D} 1_{\{X_t = o\}} dt$:

  $z_i = \sum_j p_{ij}z_j$, $i \notin D, i \neq k$; $z_o = \frac{1}{q_o} + \sum_i p_{oi}z_i$; $z|_D = 0$.

§2.4 常返

- 常返: $P_i(\forall t, \exists s > t, \text{ s.t. } X_s = i) = 1.$
- 命题2.4.5. i 常返的等价条件:

  (a) $q_i = 0$ 或$P_i(\sigma_i < \infty) = 1$,

  (b) 嵌入链$\{\hat{X}_n\}$ 常返,

  (c) 格林函数发散, 即$G_{ii} = \int p_{ii}(t)dt = \infty$,

  (d) 骨架链$\{X_{n\delta} : n = 0, 1, 2 \cdots\}$ 常返.

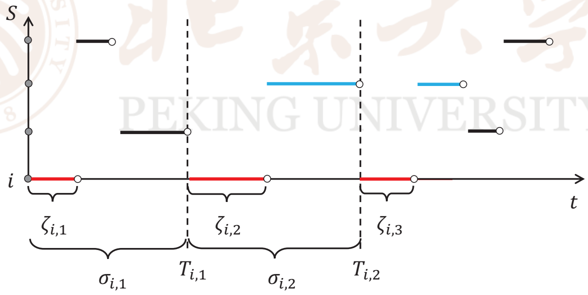
- 推论2.4.3 & 2.4.4. 在i 的总耗时:

$$\zeta_1 + \cdots + \zeta_{\hat{V}_i}, \qquad \zeta_n \text{ 独立地} \sim \mathrm{Exp}(q_i).$$

因此, 常返$\Rightarrow$ 非爆炸, $G_{ij} = \frac{1}{q_j}\hat{G}_{ij}$.

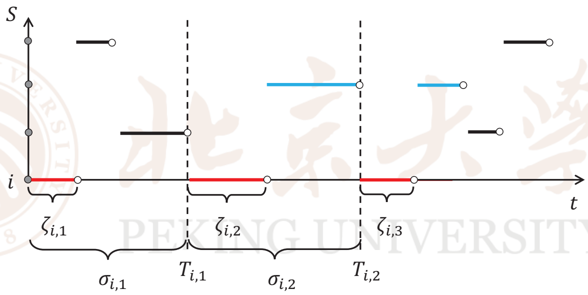
§2.5 不变分布与正常返

- 定义2.5.1. 不变分布/不变测度: $\pi = \pi\mathbf{P}(t)$, $\forall t$.
- 引理2.5.3. 若不变分布存在, 则非爆炸.
- 强马氏性(引理2.5.5): 假设i 常返, $q_i > 0$. 假设$X_0 = i$. 那么, 游弋i.i.d., 从而$(\sigma_{i,n}, \zeta_{i,n})$, $n = 1, 2, \cdots$ i.i.d.



- 命题2.5.8. 假设i 常返且$q_i > 0$. 令$V_i(t) = \int_0^t 1_{\{X_s=i\}} ds$. 则

$$P\left(\lim_{t\to\infty} \frac{1}{t} V_i(t) = \frac{1}{q_i E_i \sigma_i}\right) = 1.$$



- 访问频率: $\approx \frac{\zeta_1+\ldots+\zeta_r}{T_r} \to \frac{E\zeta}{E_o\sigma_o}$,

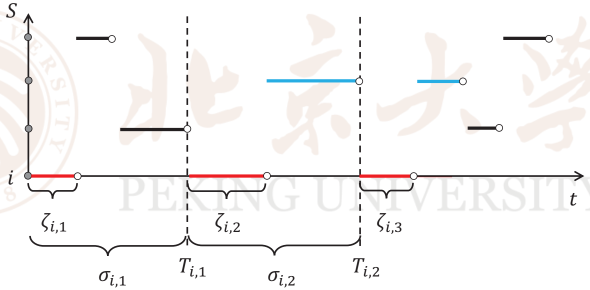
$$E\zeta = E_o \int_0^{\sigma_o} 1_{\{X_t=i\}} dt =: \mu_i.$$

命题 (命题2.5.8)

设不可约、常返. 则$q_i\mu_i = \hat{\mu}_i$; $\mu \mathbf{Q} = 0$; $\lambda \mathbf{Q} = 0$ *iff* $\lambda = c\mu$.

- $\mu_i = E_o\int_0^\sigma 1_{\{X_t=i\}}dt = \frac{1}{q_i} \times E_o V_i^{(Y)}$, $\forall i$. $\sigma = \sigma_o$.

  $\star\star = \zeta_1 + \ldots + \zeta_{\tilde{V}_i}$, $\zeta_n \sim \text{Exp}(q_i)$.



- $\frac{1}{q_i} q_{ij} = \hat{p}_{ij}$, $\lambda \mathbf{Q} = 0$ iff

$$(\lambda_j q_j) = \sum_{i \neq j} \lambda_i q_{ij} = \sum_{i \neq j} (\lambda_i q_i)\hat{p}_{ij},\text{, i.e., } q\lambda = c\hat{\mu}.$$

命题 (命题2.5.9, 推论2.5.10)

假设不可约、常返. 那么, μ 是“唯一的”不变测度.

- 记$\sigma = \sigma_o$. $\mu_i = E_o \int_0^\infty 1_{\{X_t = i, t < \sigma\}} dt = P_o(X_t = i, t < \sigma)$.
- $\sum_i \mu_i p_{ij}(s)$
  $= \int_0^\infty \sum_i P_o(X_t = i, t < \sigma) P(X_{t+s} = j | X_t = i, t < \sigma) dt$
- $= \int_0^\infty \sum_i P_o(X_t = i, t < \sigma, X_{t+s} = j) dt$
  $= \int_0^\infty P_o(t < \sigma, X_{t+s} = j) dt$
- $= E_o \int_0^\sigma 1_{\{X_{t+s} = j\}} dt = E_o \int_s^{\sigma+s} 1_{\{X_t = j\}} dt.$
- $E_0 \int_0^s 1_{\{X_t = j\}} dt = E_0 \int_\sigma^{\sigma+s} 1_{\{X_t = j\}} dt.$
- λ 是不变测度, 则也是骨架链$\{X_{n\delta} : n = 0, 1, 2, \cdots\}$ 的不变测度, 故$\lambda = c\mu$.

- 定义2.5.7. 正常返: $q_i = 0$ 或$E_i\sigma_i < \infty$.
  零常返: $q_i > 0$, $P_i(\sigma_i < \infty) = 1$, 且$E_i\sigma_i = \infty$.
- 命题2.5.11. 设不可约. 则下面三条等价: (i) 所有状态正常返, (2) 存在正常返态, (3) 存在不变分布. 此时, $\pi\mathbf{Q} = 0$,

$$\pi_i = \frac{1}{q_i E_i \sigma_i} = \frac{1}{E_o \sigma_o} E_o \int_0^{\sigma_o} 1_{\{X_t = i\}} dt \approx i \text{ 的频率}.$$

- 例2.5.4. $\pi\mathbf{Q} = 0$, 但π 不是不变分布.
- 遍历(定理2.5.17): 不可约、正常返, $\sum_i \pi_i |f(i)| < \infty$, 则

$$P_\mu \left(\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_s) ds = \sum_{i\in S} \pi_i f(i) \right) = 1.$$

- 强遍历(定理2.5.18): $\lim_{t\to\infty} p_{ij}(t) = \pi_j$, $\forall j$.

§2.6 可逆分布

- 总假设Q 不可约. 若π 为测度, 满足$\pi\mathbf{Q}=0$, 令

$$\tilde{q}_{ij} := \frac{\pi_j q_{ji}}{\pi_i}.$$

则, $\pi_i\tilde{q}_{ij}=\pi_j q_{ji}$, $\forall i,j$.

- $\tilde{\mathbf{Q}}=(\tilde{q}_{ij})_{S\times S}$ 仍为转移矩阵, 且$\tilde{q}_i=q_i$, $\pi\tilde{\mathbf{Q}}=0$.
- 命题2.6.1. $\pi_i\tilde{p}_{ij}(t)=\pi_j p_{ji}(t)$, $\forall i,j$.
- 推论2.6.2. 以下三条等价. (1) π 为不变分布, (2) $\mathbf{Q}$ 非爆炸, 且$\pi\mathbf{Q}=0$, (3) $\tilde{\mathbf{Q}}$ 非爆炸, 且$\pi\mathbf{Q}=0$.
- 推论2.6.4. 假设不可约、非爆炸, 细致平衡条件成立: $\pi_i q_{ij}=\pi_j q_{ji}$, $\forall i,j$. 那么, $\{X_t\}$ 为可逆过程.