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The growth of relative wealth and the Kelly criterion

Andrew W. Lo[1,2,3] · **H. Allen Orr**[4] · **Ruixun Zhang**[2]

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Abstract We propose an evolutionary framework for optimal portfolio growth theory in which investors subject to environmental pressures allocate their wealth between two assets. By considering both absolute wealth and relative wealth between investors, we show that different investor behaviors survive in different environments. When investors maximize their relative wealth, the Kelly criterion is optimal only under certain conditions, which are identified. The initial relative wealth plays a critical role in determining the deviation of optimal behavior from the Kelly criterion regardless of whether the investor is myopic across a single time period or maximizing wealth over an infinite horizon. We relate these results to population genetics, and discuss testable consequences of these findings using experimental evolution.

Keywords Kelly criterion · Portfolio optimization · Adaptive Markets Hypothesis · Evolutionary game theory

✉ Andrew W. Lo
alo-admin@mit.edu

H. Allen Orr
allen.orr@rochester.edu

Ruixun Zhang
zhangruixun@gmail.com

1 Sloan School of Management, Massachusetts Institute of Technology, 100 Main Street E62-618, Cambridge, MA 02142-1347, USA

2 Laboratory for Financial Engineering, Massachusetts Institute of Technology, One Broadway, E70-1375, Cambridge, MA 02142, USA

3 Computer Science and Artificial Intelligence Laboratory, Massachusetts Institute of Technology, 32 Vassar Street, Cambridge, MA 02139, USA

4 Department of Biology, University of Rochester, Hutchison 342, Rochester, NY 14627, USA

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1 Introduction

The allocation of wealth among alternative assets is one of an individual's most important financial decisions. The groundbreaking work of Markowitz (1952) in mean-variance theory, used to analyze asset allocation, has remained the cornerstone of modern portfolio theory. This has led to numerous breakthroughs in financial economics, including the famous Capital Asset Pricing Model (Sharpe 1964; Lintner 1965a, b; Treynor 1965; Mossin 1966). The influence of this paradigm goes far beyond academia, however. For example, it has become an integral part of modern investment management practice (Reilly and Brown 2011).

More recently, economists have also adopted the use of evolutionary principles to understand economic behavior, leading to the development of evolutionary game theory (Maynard Smith 1982); the evolutionary implications of probability matching (Cooper and Kaplan 1982), group selection (Zhang et al. 2014a), and cooperation and altruism (Alexander 1974; Hirshleifer 1977, 1978); and the process of selection of firms (Luo 1995) and traders (Blume and Easley 1992; Kogan et al. 2006a; Hirshleifer and Teoh 2009). The Adaptive Markets Hypothesis (Lo 2004, 2017) provides economics with a more general evolutionary perspective, reconciling economic theories based on the Efficient Markets Hypothesis with behavioral economics. Under this hypothesis, the neoclassical models of rational behavior coexist with behavioral models, and what had previously been cited as counterexamples to rationality—loss aversion, overconfidence, overreaction, and other behavioral biases—become consistent with an evolutionary model of human behavior.

The evolutionary perspective brings new insights to economics from beyond the traditional neoclassical realm, helping to reconcile inconsistencies in behavior between *Homo economicus* and *Homo sapiens* (Kahneman and Tversky 1979; Brennan and Lo 2011). In particular, evolutionary models of behavior provide important insights into the biological origin of time preference and utility functions (Rogers 1994; Waldman 1994; Samuelson 2001; Zhang et al. 2014b), in fact justifying their existence, and allow us to derive conditions about their functional form (Hansson and Stuart 1990; Robson 1996, 2001a) (see Robson 2001b; Robson and Samuelson 2009 for comprehensive reviews of this literature). In addition, the experimental evolution of biological organisms has been suggested as a novel approach to understanding economic preferences, given that it allows the empirical study of preferences by placing organisms in specifically designed environments (Burnham et al. 2015).

The evolutionary approach to investing is closely related to optimal portfolio growth theory, as explored by Kelly (1956), Hakansson (1970), Thorp (1971), Algoet and Cover (1988), Browne and Whitt (1996), and Aurell et al. (2000), among others. While the evolutionary framework tends to focus on the long-term performance of a strategy, investors are also concerned with the short to medium term (Browne 1999). Myopic investor behavior has been documented in both theoretical and empirical studies (Strotz 1955; Stroyan 1983; Thaler et al. 1997; Bushee 1998). Since much of the field of population genetics focuses on short-term competition between different

types of individuals, population geneticists have applied their ideas to portfolio theory (Frank 1990, 2011; Orr 2017), in some cases considering the maximization of one-period expected wealth.

The market dynamics of investment strategies under evolutionary selection have been explored under the assumption that investors will try to maximize absolute wealth (Evstigneev et al. 2002; Amir et al. 2005; Hens and Schenk-Hoppé 2005; Evstigneev et al. 2006). Some studies have found that individual investors with more accurate beliefs will accumulate more wealth, and thus dominate the economy (Sandroni 2000, 2005), while others have argued that wealth dynamics need not lead to rules that maximize expected utility using rational expectations (Blume and Easley 1992), and that investors with incorrect beliefs may drive out those with correct beliefs (Blume and Easley 2006). Research on the performance of rational versus irrational traders has also adopted evolutionary ideas; for example, it has been shown that irrational traders can survive in the long run, resulting in prices that diverge from fundamental values (Long et al. 1990, 1991; Biais and Shadur 2000; Hirshleifer and Luo 2001; Hirshleifer et al. 2006; Kogan et al. 2006b; Yan 2008).

Traditional portfolio growth theory has focused on absolute wealth and the Kelly criterion (Kelly 1956; Thorp 1971). Instead of studying the growth of absolute wealth, however, we will consider the relative wealth in the spirit of Orr (2017). Relative wealth or income has been discussed in a number of studies (Robson 1992; Bakshi and Chen 1996; Corneo and Jeanne 1997; Hens and Schenk-Hoppé 2005; Frank 2011), and the behavioral economics literature provides voluminous evidence that investors sometimes assess their performance relative to a reference group (Frank 1985; Clark and Oswald 1996; Clark et al. 2008). It is particularly important to understand the consequences of investment decisions in a setting where relative wealth is the standard, not absolute wealth. As Burnham et al. (2015) pointed out, "if people are envious by caring about relative wealth, then free trade may make all parties richer, but may cause envious people to be less happy. If economics misunderstands human nature, then free trade may simultaneously increase wealth and unhappiness."

In this paper, we compare the implications of maximizing relative wealth to maximizing absolute wealth over both short-term and long-term investment horizons. We use ideas from Orr (2017), and compare his results to an extension of the binary choice model of Brennan and Lo (2011). We consider two assets in a discrete-time model, and an investor who allocates her wealth between the two assets. Rather than maximizing her absolute wealth, the investor maximizes her wealth relative to another investor with a fixed behavior. We consider the cases of one time period, multiple periods, and an infinite time horizon. We then ask the question: what is the optimal behavior for an investor as a function of the environment, given that the environment consists of the asset returns and the behavior of the other participants? In our approach, we define relative wealth as a proportion of the total wealth, which corresponds closely to the allele frequency in population genetics. This analogy acts as a bridge to earlier literature on the relevance of relative wealth to behavior. While some of our results will be familiar to population geneticists, they do not appear to be widely known in a financial context. For completeness, we derive them from first principles in this new context.

Our approach leads to several interesting conclusions about the Kelly criterion. We show that it is the optimal behavior if the investor maximizes her absolute wealth in the

case of an infinite horizon (see also Brennan and Lo 2011). In the case that the investor maximizes her relative wealth, we identify the conditions under which it is optimal, and the conditions under which the investor should deviate from it. The investor's initial relative wealth—which represents the investor's market power—plays a critical role. Moreover, the dominant investor's optimal behavior is different from the minorant investor's optimal behavior.

In Sect. 2 of this paper, we consider a two-asset model in which investors maximize their absolute wealth. It is shown that the long-run optimal behavior is equivalent to the behavior implied by the Kelly criterion. Section 3 extends the binary choice model, and considers in a non-game-theoretic framework the case of two investors who maximize their wealth relative to the population, given the other investor's behavior. The Kelly criterion emerges as a special case under certain environmental conditions. Section 4 provides a numerical example to illustrate the theoretical results. Section 5 discusses several implications which can be tested through experimental evolutionary techniques. We end with a discussion in Sect. 6, and provide proofs in Appendix A.

2 Maximizing absolute wealth: the Kelly criterion

Consider two assets, a and b, in a discrete-time model, each generating gross returns $X_a \in (0, \infty)$ and $X_b \in (0, \infty)$ in one period. For example, asset a can be a risky asset whereas asset b can be the riskless asset. In this case, $X_a \in (0, \infty)$ and $X_b = 1 + r$, where r is the risk-free interest rate. In general, $(X_{a,t}, X_{b,t})$ are IID over time $t = 1, 2, \ldots$, and are described by the probability distribution function $\Phi(X_a, X_b)$.

Consider an investor who allocates $f \in [0, 1]$ of her wealth in asset a and $1 - f$ in asset b. We will refer to f as the investor's behavior henceforth. We assume that:

Assumption 1 (X_a, X_b) and $\log(fX_a + (1-f)X_b)$ have finite moments up to order 2 for all $f \in [0, 1]$.

Note that Assumption 1 guarantees that the gross return of any investment portfolio is positive. In other words, the investor cannot lose more than what she has. This is made possible by assuming that X_a and X_b are positive and f is between 0 and 1. In other words, the investor only allocates her money between two assets, and no short-selling or leverage is allowed.[1]

Let n_t^f be the total wealth of investor f in period t. To simplify notation, let $\omega_t^f = fX_{a,t} + (1-f)X_{b,t}$ be the gross return of investor f's portfolio in period t. With these notational conventions in mind, the portfolio growth from period $t-1$ to period t is:

$$n_t^f = n_{t-1}^f \left(fX_{a,t} + (1-f)X_{b,t}\right) = n_{t-1}^f \omega_t^f .$$

[1] One could relax this assumption by allowing short-selling and leverage, which corresponds to $f < 0$ or $f > 0$. However, f still needs to be restricted such that $fX_a + (1-f)X_b$ is always positive. This does not change our results in any essential way, but it will complicate the presentation of some results mathematically. Therefore, we stick to the simple assumption that $f \in [0, 1]$ as in Brennan and Lo (2011).

Through backward recursion, the total wealth of investor f in period T is given by

$$n_T^f = \prod_{t=1}^{T} \omega_t^f = \exp\left(\sum_{t=1}^{T} \log \omega_t^f\right).$$

Taking the logarithm of wealth and applying Kolmogorov's law of large numbers, we have:

$$\frac{1}{T} \log n_T^f = \frac{1}{T} \sum_{t=1}^{T} \log \omega_t^f \xrightarrow{p} \mathbb{E}\left[\log \omega_t^f\right] = \mathbb{E}\left[\log\left(f X_a + (1-f) X_b\right)\right], \quad (1)$$

as T increases without bound, where "$\xrightarrow{p}$" in (1) denotes convergence in probability. We have assumed that $n_0^f = 1$ without loss of generality.

The expression (1) is simply the expectation of the log-geometric-average growth rate of investor f's wealth, and we will call it $\mu(f)$ henceforth:

$$\mu(f) = \mathbb{E}\left[\log\left(f X_a + (1-f) X_b\right)\right]. \quad (2)$$

The optimal f that maximizes (2) is given by

Proposition 1 *The optimal allocation f^{Kelly} that maximizes investor f's absolute wealth as T increases without bound is*

$$f^{Kelly} = \begin{cases} 1 & \text{if } \mathbb{E}[X_a/X_b] > 1 \text{ and } \mathbb{E}[X_b/X_a] < 1, \\ \text{solution to (4)} & \text{if } \mathbb{E}[X_a/X_b] \geq 1 \text{ and } \mathbb{E}[X_b/X_a] \geq 1, \\ 0 & \text{if } \mathbb{E}[X_a/X_b] < 1 \text{ and } \mathbb{E}[X_b/X_a] > 1, \end{cases} \quad (3)$$

where f^{Kelly} is defined implicitly in the second case of (3) by:

$$\mathbb{E}\left[\frac{X_a - X_b}{f^{Kelly} X_a + (1 - f^{Kelly}) X_b}\right] = 0. \quad (4)$$

The optimal allocation given in Proposition 1 coincides with the Kelly criterion (Kelly 1956; Thorp 1971) in probability theory and the portfolio choice literature. To emphasize this connection, we refer to this optimal allocation as the Kelly criterion henceforth. As we will see, in the case of maximizing an individual's relative wealth, the Kelly criterion plays a key role as a reference strategy.

In portfolio theory, the Kelly criterion is used to determine the optimal size of a series of bets in the long run. Although this strategy's promise of doing better than any other strategy in the long run seems compelling, some researchers have argued against it, principally because the specific investing constraints of an individual may override the desire for an optimal growth rate. In other words, different investors might have different utility functions. In fact, to an individual with logarithmic utility, the Kelly criterion will maximize the expected utility, so the Kelly criterion can be considered as the optimal strategy under expected utility theory with a specific utility function.

3 Maximizing relative wealth

In this section, we consider two investors. The first investor allocates $f \in [0,\ 1]$ of her wealth in asset a and $1 - f$ in asset b. The second investor allocates $g \in [0,\ 1]$ of his wealth in asset a and $1 - g$ in asset b. Investor f's objective is to maximize the proportion of her wealth relative to the total wealth in the population, which we define as investor f's relative wealth. Note that we use f and g to mean the proportion of wealth and as a label for the investor, to simplify notation.

Here we can introduce a concept taken from evolutionary theory. In population genetics, the metric for natural selection is the expected reproduction of a genotype divided by the average reproduction of the population, i.e., the relative reproduction, analogous to investor f's relative wealth. Our consideration of the relative wealth rather than the absolute wealth naturally unlocks existing tools and ideas from population genetics for us.

In the case of maximizing relative wealth, the initial wealth plays an important role in the optimal allocation. Let $\lambda \in (0,\ 1)$ be the relative initial wealth of investor f:

$$\lambda = \frac{n_0^f}{n_0^f + n_0^g}.$$

Let q_t^f be the relative wealth of investor f in subsequent periods $t = 1,\ 2, \ldots$ q_t^f and q_t^g are defined similarly:

$$\begin{aligned} q_t^f &= \frac{n_t^f}{n_t^f + n_t^g} = \frac{1}{1 + n_t^g / n_t^f}, \\ q_t^g &= 1 - q_t^f. \end{aligned}$$

It is obvious that the ratio n_t^g / n_t^f is sufficient to determine the relative wealth q_t^f. Let R_T^f be the T-period average log-relative-growth:

$$R_T^f = \frac{1}{T} \log \frac{\prod_{t=1}^T \omega_t^g}{\prod_{t=1}^T \omega_t^f} = \frac{1}{T} \sum_{t=1}^T \log \frac{\omega_t^g}{\omega_t^f}. \tag{5}$$

Then we can write the relative wealth in period T as:

$$q_T^f = \frac{1}{1 + \frac{n_T^g}{n_T^f}} = \frac{1}{1 + \frac{(1-\lambda)\prod_{t=1}^T \omega_t^g}{\lambda \prod_{t=1}^T \omega_t^f}} = \frac{1}{1 + \frac{1-\lambda}{\lambda} \exp\left(T R_T^f\right)}. \tag{6}$$

Analogs to Eqs. (5)–(6) are well known in the population genetics literature, used when the fitnesses of genotypes are assumed to vary randomly through time. (For reviews of this literature, see Felsenstein 1976 and Gillespie 1991, Chap. 4.)

3.1 One-period results

We first consider a myopic investor, who maximizes her expected relative wealth in the first period. By (6), the expectation of q_1^f is:

$$\mathbb{E}\left[q_1^f\right] = \mathbb{E}\left[\frac{1}{1+\frac{1-\lambda}{\lambda}\frac{\omega^g}{\omega^f}}\right].$$

Here we have dropped the subscripts in ω_1^f and ω_1^g, and instead simply use ω^f and ω^g, because there is only one period to consider.

Given investor g, we denote f_1^* as investor f's optimal allocation that maximizes $\mathbb{E}[q_1^f]$. There is no general formula to compute $\mathbb{E}[q_1^f]$ because it involves the expectation of the ratio of random variables. Population geneticists sometimes use diffusion approximations to estimate similar quantities, for example, the change in allele frequency (Gillespie 1977; Frank and Slatkin 1990; Frank 2011), which are essentially linear approximations of the nonlinear quantity using the Taylor series. The diffusion approximation is also used by Orr (2017) in a similar model for relative wealth.

Without the diffusion approximation, one can still characterize f_1^* to a certain degree:

Proposition 2 *The optimal behavior of investor f that maximizes expected relative wealth in the first period is given by*:

$$f_1^* = \begin{cases} 1 & \text{if } \mathbb{E}\left[\frac{(X_a-X_b)\omega^g}{(\lambda X_a+(1-\lambda)\omega^g)^2}\right] > 0, \\ \text{solution to (8)} & \text{if } \mathbb{E}\left[\frac{(X_a-X_b)\omega^g}{(\lambda X_a+(1-\lambda)\omega^g)^2}\right] \le 0 \text{ and } \mathbb{E}\left[\frac{(X_a-X_b)\omega^g}{(\lambda X_b+(1-\lambda)\omega^g)^2}\right] \ge 0, \\ 0 & \text{if } \mathbb{E}\left[\frac{(X_a-X_b)\omega^g}{(\lambda X_b+(1-\lambda)\omega^g)^2}\right] < 0, \end{cases} \tag{7}$$

where f_1^* is defined implicitly in the second case of (7) by:

$$\mathbb{E}\left[\frac{(X_a - X_b)\omega^g}{(\lambda\omega^f + (1-\lambda)\omega^g)^2}\right] = 0. \tag{8}$$

In general, the optimal behavior f_1^* is a function of g. The next proposition asserts that f_1^* is always "bounded" by g.

Proposition 3 *To maximize the expected relative wealth in period 1, investor f should never deviate more from the Kelly criterion f^{Kelly} than investor g in the same direction*:

- *If $g = f^{Kelly}$, then $f_1^* = f^{Kelly}$.*
- *If $g < f^{Kelly}$, then $f_1^* > g$.*
- *If $g > f^{Kelly}$, then $f_1^* < g$.*

The conclusion in Proposition 3 makes intuitive sense. When investor g takes a position that is riskier than the Kelly criterion, investor f should never be even riskier

than investor g. Similarly, when investor g takes a position that is more conservative than the Kelly criterion, investor f should never be even more conservative than investor g.

It is interesting to compare the optimal behavior f_1^* with the Kelly criterion f^{Kelly}, which is provided in the next proposition. It shows that when g is not far from the Kelly criterion, the relationship between f_1^* and f^{Kelly} depends on the initial relative wealth of investor f.

Proposition 4 *If investor f is the dominant investor $\left(\lambda > \frac{1}{2}\right)$, then she should be locally more/less risky than Kelly in the same way as investor g: for small $\epsilon > 0$,*

$$g = f^{Kelly} - \epsilon \Rightarrow f_1^* < f^{Kelly},$$
$$g = f^{Kelly} + \epsilon \Rightarrow f_1^* > f^{Kelly}.$$

If investor f is the minorant investor $\left(\lambda < \frac{1}{2}\right)$, then she should be locally more/less risky than Kelly in the opposite way as investor g: for small $\epsilon > 0$,

$$g = f^{Kelly} - \epsilon \Rightarrow f_1^* > f^{Kelly},$$
$$g = f^{Kelly} + \epsilon \Rightarrow f_1^* < f^{Kelly}.$$

If investor f starts with the same amount of wealth as investor g $\left(\lambda = \frac{1}{2}\right)$, then she should be locally Kelly:

$$g \approx f^{Kelly} \Rightarrow f_1^* \approx f^{Kelly}.$$

Note that when g is far from the Kelly criterion, the conclusions in Proposition 4 may not necessarily hold. Section 4 provides a numerical example (see Fig. 1b) where investor f is the minorant investor $\left(\lambda < \frac{1}{2}\right)$ and $g \ll f^{Kelly}$, but $f_1^* < f^{Kelly}$. However, Orr (2017) has shown that these results are still approximately true for any g up to a diffusion approximation, which is consistent with the numerical results for maximizing one-period relative wealth in Fig. 1a. We will provide more discussion on this point in Sect. 4.

3.2 Multi-period results

The previous results are based on maximizing the expected relative wealth in period 1: $\mathbb{E}[q_1^f]$. To generalize these results to maximizing expected relative wealth in period T: $\mathbb{E}[q_T^f]$, we have:

Proposition 5 *The optimal behavior of investor f that maximizes expected relative wealth in the Tth period is given by*:

$$f_T^* = \begin{cases} 1 & \text{if } \mathbb{E}\left[\frac{\exp(TR_T^1)\left(T-\sum_{t=1}^T \frac{X_{bt}}{X_{at}}\right)}{\left(1+\frac{1-\lambda}{\lambda}\exp(TR_T^1)\right)^2}\right] > 0, \\ \text{solution to (10)} & \text{if } \mathbb{E}\left[\frac{\exp(TR_T^1)\left(T-\sum_{t=1}^T \frac{X_{bt}}{X_{at}}\right)}{\left(1+\frac{1-\lambda}{\lambda}\exp(TR_T^1)\right)^2}\right] \leq 0 \text{ and } \mathbb{E}\left[\frac{\exp(TR_T^0)\left(\sum_{t=1}^T \frac{X_{at}}{X_{bt}}-T\right)}{\left(1+\frac{1-\lambda}{\lambda}\exp(TR_T^0)\right)^2}\right] \geq 0, \\ 0 & \text{if } \mathbb{E}\left[\frac{\exp(TR_T^0)\left(\sum_{t=1}^T \frac{X_{at}}{X_{bt}}-T\right)}{\left(1+\frac{1-\lambda}{\lambda}\exp(TR_T^0)\right)^2}\right] < 0, \end{cases} \tag{9}$$

where f_1^* *is defined implicitly in the second case of* (9) *by*:

$$\mathbb{E}\left[\frac{\exp(TR_T^f)\sum_{t=1}^T \frac{X_{at}-X_{bt}}{fX_{at}+(1-f)X_{bt}}}{\left(1+\frac{1-\lambda}{\lambda}\exp(TR_T^f)\right)^2}\right] = 0. \tag{10}$$

We have assumed that f is constant through time, which implies that the investor does not dynamically change her position from period to period. This passive strategy is of interest for two reasons: the information about each investor's relative wealth may be difficult to get in each period, and it is also costly to rebalance the portfolio after each period.

If the investor is indeed able to adjust f dynamically at each new period as a function of her current relative wealth, it is clear the expected growth of her relative wealth can be increased. This is studied numerically in a similar model in Orr (2017).

Similarly, one can "bound" f_T^* by g, and compare f_T^* with f^{Kelly} when g is near the Kelly criterion.

Proposition 6 *The conclusions in Propositions 3 and 4 hold for* f_T^* *in multi-period,* $T = 2, 3, \ldots$

Simply put, when investor g takes a position that is riskier than the Kelly criterion, investor f should never be even riskier than investor g, no matter how many horizons forward she is looking. Similarly, when investor g takes a position that is more conservative than the Kelly criterion, investor f should never be even more conservative than investor g, no matter how many horizons forward she is looking. On the other hand, if investor g deviates from the Kelly criterion only slightly, then investor f should deviate from the Kelly criterion in the opposite direction than investor g, provided that she has less initial wealth than investor g, but in the same direction, provided that she has more initial wealth than investor g.

As a special case, if investor g is playing the Kelly criterion strategy, then investor f should also play Kelly; if investor g is not playing Kelly, however, investor f should also not play Kelly. This implies that the Kelly criterion is a Nash equilibrium when investors maximize their relative wealth. This is consistent with the existing literature of evolutionary portfolio theory (Evstigneev et al. 2002, 2006) when absolute wealth is maximized.

Note that in the multi-period case, one does not have analogous results from a diffusion approximation as in the one-period case (Orr 2017). The numerical results

of Sect. 4 show that the condition that g is near the Kelly criterion is essential (see Fig. 2a).

3.3 Infinite horizon

Recall from (5) that the T-period average log-relative-growth R_T^f is given by:

$$R_T^f = \frac{1}{T}\sum_{t=1}^{T} \log \omega_t^g - \frac{1}{T}\sum_{t=1}^{T} \log \omega_t^f \xrightarrow{p} \mu(g) - \mu(f), \tag{11}$$

as T increases without bound. It is therefore easy to see from (6) that:

Proposition 7 *As T increases without bound, the relative wealth of investor f converges in probability to a constant*:

$$q_T^f \xrightarrow{p} \begin{cases} 0 \text{ if } \mu(f) < \mu(g), \\ \lambda \text{ if } \mu(f) = \mu(g), \\ 1 \text{ if } \mu(f) > \mu(g). \end{cases} \tag{12}$$

Proposition 7 is consistent with well-known results in the population genetics literature (see Gillespie 1973, for example) as well as in the behavioral finance literature, as in Brennan and Lo (2011). It asserts that investor f's relative wealth will converge to 1 as long as its log-geometric-average growth rate $\mu(f)$ is greater than investor g's. This implies that when T increases without bound, there are multiple behaviors that are all optimal in the following sense:

$$\begin{aligned} \arg\max_f \lim_{T\to\infty} q_T^f &= \arg\max_f \mathbb{E}\left[\lim_{T\to\infty} q_T^f\right] \\ &= \arg\max_f \lim_{T\to\infty} \mathbb{E}\left[q_T^f\right] = \{f : \mu(f) > \mu(g)\}. \end{aligned}$$

Note that the above equality uses the dominant convergence theorem (q_T^f is always bounded) to switch the limit and the expectation operator.

However, this is not equivalent to the limit of the optimal behavior f_T^* as T increases without bound because one cannot switch the operator "arg max" and "lim" in general, and

$$\arg\max_f \lim_{T\to\infty} \mathbb{E}\left[q_T^f\right] \neq \lim_{T\to\infty} \arg\max_f \mathbb{E}\left[q_T^f\right].$$

In fact, Sect. 4 provides such an example.

4 A numerical example

We construct a numerical example in this section to illustrate the results of Sects. 2 and 3. Consider the following two simple assets:

$$X_a = \begin{cases} \alpha \text{ with probability } p, \\ \beta \text{ with probability } 1-p, \end{cases} \qquad X_b = \gamma \quad \text{with probability 1.}$$

In this case, asset a is risky and asset b is riskless. The expected relative wealth of investor f in period T is explicitly given by:

$$\mathbb{E}\left[q_T^f\right] = \sum_{k=0}^{T} \frac{\binom{T}{k} p^k (1-p)^{T-k}}{1 + \frac{1-\lambda}{\lambda} \exp\left(k \log \frac{g\alpha+(1-g)\gamma}{f\alpha+(1-f)\gamma} + (T-k) \log \frac{g\beta+(1-g)\gamma}{f\beta+(1-f)\gamma}\right)}. \tag{13}$$

It is easy to numerically solve from (13) the optimal behavior f_T^* for any given environment $\alpha,\ \beta,\ \gamma,\ p$.

For simplicity, we focus on one particular environment henceforth:

$$X_a = \begin{cases} 2 & \text{with probability } 0.5, \\ 0.5 & \text{with probability } 0.5, \end{cases} \qquad X_b = 1 \text{ with probability 1.}$$

It is easy to show by Proposition 1 that $f^{Kelly} = \frac{1}{2}$ in this case. As noted following Assumption 1, to guarantee that the gross return for any investment portfolio is positive, f can take values between -1 and 2. For simplicity and consistency with the theoretical results, we will restrict f to be between 0 and 1, which does not affect the comparisons below in any essential way.

4.1 Maximizing one-period relative wealth

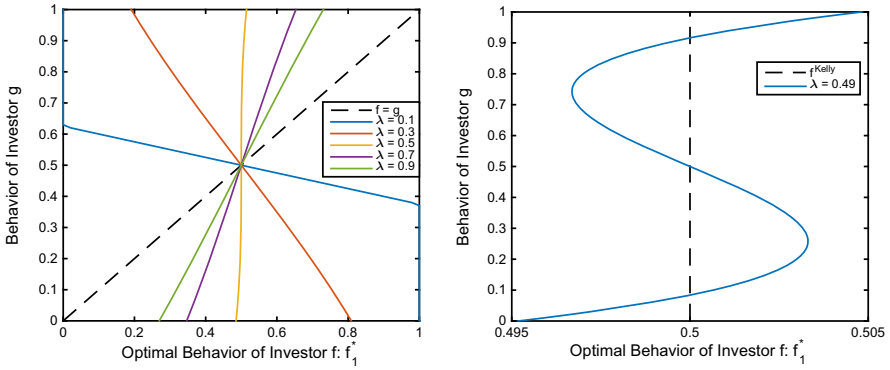
We first consider the case of maximizing one's relative wealth in period 1. Figure 1a shows f_1^* for several different cases of investor f's relative wealth λ. We can see that investor f's optimal behavior is always "bounded" by g, and the more dominant that investor f is, the closer f_1^* will be to g. These observations are consistent with Propositions 3 and 4.

Figure 1b zooms into one particular case of $\lambda = 0.49$, with the f-axis from 0.495 to 0.505. It emphasizes the fact that the comparison between f_1^* and f^{Kelly} is only valid when g is close to the Kelly criterion, as asserted in Proposition 4. However, except for this particular case, the conclusions in Proposition 4 are true for any g. This provides numerical evidence that the diffusion approximation in Orr (2017) is relatively accurate for one-period results.

4.2 Maximizing multi-period relative wealth

Next we consider maximizing relative wealth over multiple periods. Figure 2 shows the evolution of f_T^* for three different initial values of relative wealth, $\lambda = 0.2,\ 0.5,\ 0.8$. It is clear that investor f's optimal behavior is always "bounded" by g as T increases. In this example, it is also clear that f_T^* does not converge to f^{Kelly} as T increases without bound.

When investor f is the minorant investor (Fig. 2a, $\lambda = 0.2$), her optimal behavior deviates from the Kelly criterion in the opposite direction of investor g near $g = 0.5$.



(a) Comparison of different initial relative wealth **(b)** Comparison with Kelly criterion

Fig. 1 Optimal behavior of investor f as a function of g: f_1^*. Different values of λ correspond to different initial levels of relative wealth. **a** Investor f's optimal behavior is always "bounded" by g, and the more dominant investor f is, the closer f_1^* is to g. **b** One particular example that the comparison between f_1^* and f^{Kelly} in Proposition 4 is only valid when g is close to the Kelly criterion

When investor f is the dominant investor (Fig. 2c, $\lambda = 0.8$), her optimal behavior deviates from the Kelly criterion in the same direction as investor g near $g = 0.5$. When investor f has the same initial wealth as investor g (Fig. 2b, $\lambda = 0.5$), her optimal behavior is approximately equal to the Kelly criterion near $g = 0.5$.

It is interesting to note that when investor f is the minorant investor (Fig. 2a, $\lambda = 0.2$), the comparison between f_T^* and f^{Kelly} is only true when g is close to the Kelly criterion. This is more true as the number of periods T increases. In this case, the fact that g must be close to the Kelly criterion becomes critical.

5 Testable implications

Given the similarities between biological evolution and our financial model, it should be possible to design experimental evolutionary studies to test our model's implications biologically. As Burnham et al. (2015) have pointed out, experimental evolution allows the empirical investigation of decision making under uncertainty. Central to this idea is the creation of test and control environments that vary in payoffs—or in a biological context, fitness. Various species, ranging from bacteria to *Drosophila* (fruit flies), have been used to design experiments to understand decision making under uncertainty (Mery and Kawecki 2002; Beaumont et al. 2009; Dunlap and Stephens 2014).

To test our model, one could create an environment in which *Drosophila* individuals must choose between two places to lay their eggs (media A and B). Different media would be associated with fruits with different odors, like orange and pineapple, as a signal to *Drosophila*. Competing "investment" payoffs would be implemented as different rules for harvesting *Drosophila* eggs from the two media.[2]

In principle, one could create any possible payoff through different harvesting rules. For instance, to test our numerical example, let one of the media be the safe asset,

[2] We thank Terence C. Burnham for suggesting this design.

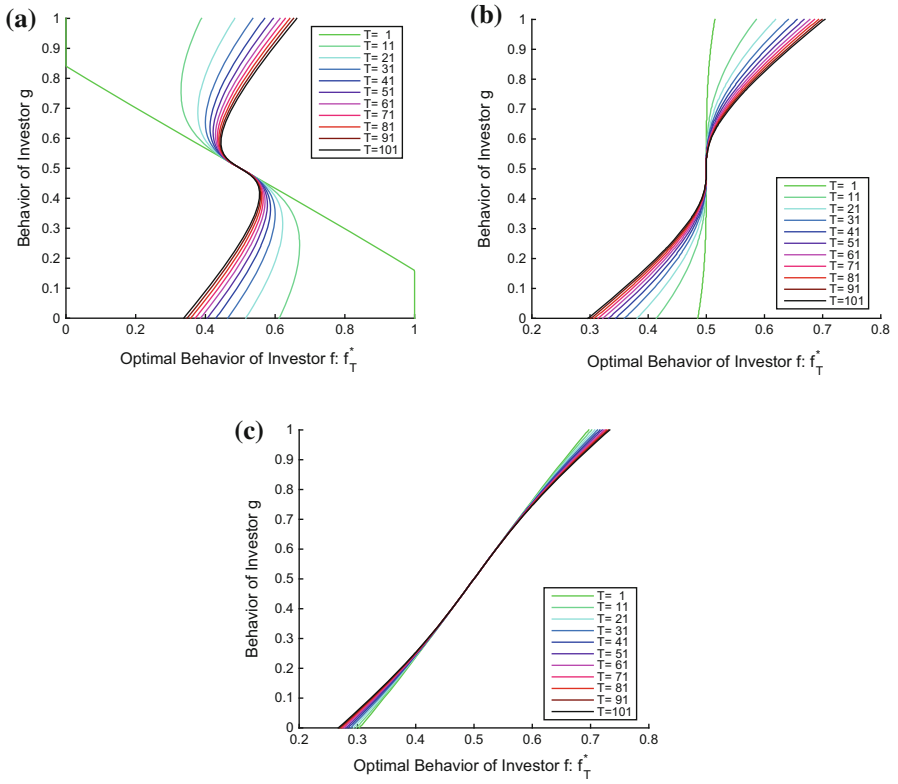


Fig. 2 Evolution of the optimal behavior of investor f: f_T^*, $T = 1, 11, \ldots, 101$. Different values of λ correspond to different initial levels of relative wealth. **a** $\lambda = 0.2$. **b** $\lambda = 0.5$. **c** $\lambda = 0.8$

and the other the risky asset. One could harvest 100 eggs every generation from the safe asset (e.g., the orange-scented medium), and then 0 or 200 eggs every generation with equal probability from the risky asset (e.g., the pineapple-scented medium). Over many generations, one would measure the percentage of eggs laid by *Drosophila* on the pineapple-scented medium, which one would treat as a proxy for the allocation to the risky asset (behavior f in our model).

The above procedure creates an "investor," and tracks the evolution of its "investing" behavior given the environment. One might use two or more distinct groups of *Drosophila* and expose them to the same reproductive environment. By varying the initial relative proportion of the two groups of *Drosophila*, one would measure the investing behavior (the percentage of eggs laid on pineapple) over many generations, as a function of the initial relative wealth, and the different payoffs of the safe and risky assets, and compare that to the predictions from the theory.

6 Discussion

Unlike the traditional theory of portfolio growth, this paper imports ideas from evolutionary biology and population genetics, focusing on the relative wealth of an investor

rather than on the absolute wealth. Relative wealth is important financially because success and satisfaction are sometimes measured by investors relative to the success of others (Robson 1992; Frank 1990, 2011; Bakshi and Chen 1996; Clark and Oswald 1996; Corneo and Jeanne 1997; Clark et al. 2008). Our model considers the case of two investors in a non-game-theoretic framework. We show how the optimal behavior of one investor is dependent on the other investor's behavior, which might be far from the Kelly criterion. While some of our results are already known in the finance literature or the population genetics literature, they are not known together in both, and therefore they are included for completeness.

We consider myopic investors who maximize their expected relative wealth over a single period, and investors who maximize their relative wealth over multiple periods. Similar consequences hold in both cases. When one investor is wealthier than the other, that investor should roughly mimic the other's behavior in being more or less aggressive than the Kelly criterion. Conversely, when one investor is poorer than the other, that investor should roughly act in the opposite manner of the other investor (Orr 2017).

As described above, it should be possible to design empirical biological studies to test the ideas of this paper. For example, one could design an experimental evolutionary study with a riskless condition (with constant fitness, corresponding to a fixed payoff) and a risky condition (with variable fitness, corresponding to different payoffs), much like the numerical example considered in Sect. 4. More generally, one could design an experimental environment with two random fitnesses that follow two different distributions. By varying the proportion of each population type exposed to each environment, one could create any type of "investor" as described in our model. Eventually, one would observe the growth of different types of "investors" to test various predictions about relative wealth in this paper.

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Appendix A: Proofs

Proof of Proposition 1 See Brennan and Lo (2011). □

Proof of Proposition 2 The first partial derivative of $\mathbb{E}[q_1^f]$ to f is:

$$\frac{\partial \mathbb{E}[q_1^f]}{\partial f} = \lambda(1-\lambda)\mathbb{E}\left[\frac{(X_a - X_b)\omega^g}{(\lambda\omega^f + (1-\lambda)\omega^g)^2}\right].$$

The second partial derivative of $\mathbb{E}[q_1^f]$ to f is:

$$\frac{\partial^2 \mathbb{E}[q_1^f]}{\partial f^2} = -2\lambda^2(1-\lambda)\mathbb{E}\left[\frac{(X_a - X_b)^2\omega^g}{(\lambda\omega^f + (1-\lambda)\omega^g)^3}\right] \leq 0,$$

which indicates that $\mathbb{E}[q_1^f]$ is a concave function of f. Therefore, it suffices to consider the value of the first partial derivative at its endpoints 0 and 1.

$$f_1^* = \begin{cases} 1 & \text{if } \frac{\partial \mathbb{E}[q_1^f]}{\partial f}\big|_{f=1} > 0, \\ 0 & \text{if } \frac{\partial \mathbb{E}[q_1^f]}{\partial f}\big|_{f=0} < 0, \\ \text{solution to } \frac{\partial \mathbb{E}[q_1^f]}{\partial f} = 0 & \text{otherwise.} \end{cases}$$

Proposition 2 follows from trivial simplifications of the above equation. □

Proof of Proposition 3 Consider $\frac{\partial \mathbb{E}[q_1^f]}{\partial f}$ when $f = g$:

$$\left.\frac{\partial \mathbb{E}[q_1^f]}{\partial f}\right|_{f=g} = \lambda(1-\lambda)\mathbb{E}\left[\frac{X_a - X_b}{fX_a + (1-f)X_b}\right].$$

Note that the righthand side consists of a factor that also appears in the first order condition (4) of the Kelly criterion. Therefore its sign is determined by whether f is larger than f^{Kelly}:

$$\left.\frac{\partial \mathbb{E}[q_1^f]}{\partial f}\right|_{f=g} \begin{cases} >0 \text{ if } f = g < f^{Kelly}, \\ =0 \text{ if } f = g = f^{Kelly}, \\ <0 \text{ if } f = g > f^{Kelly}. \end{cases} \tag{A.1}$$

Since $\mathbb{E}[q_1^f]$ is concave as a function of f for any g, we know that:

$$f_1^* \begin{cases} >g \text{ if } g < f^{Kelly}, \\ =g \text{ if } g = f^{Kelly}, \\ <g \text{ if } g > f^{Kelly}, \end{cases}$$

which completes the proof. □

Proof of Proposition 4 The cross partial derivative of $\mathbb{E}[q_1^f]$ is:

$$\frac{\partial^2 \mathbb{E}[q_1^f]}{\partial f \partial g} = \lambda(1-\lambda)\mathbb{E}\left[\frac{(X_a - X_b)^2(\lambda\omega^f - (1-\lambda)\omega^g)}{(\lambda\omega^f + (1-\lambda)\omega^g)^3}\right].$$

Consider $\frac{\partial^2 \mathbb{E}[q_1^f]}{\partial f \partial g}$ when $f = g = f^{Kelly}$:

$$\left.\frac{\partial^2 \mathbb{E}[q_1^f]}{\partial f \partial g}\right|_{f=g} = 2\lambda(1-\lambda)\left(\lambda - \frac{1}{2}\right)\mathbb{E}\left[\left(\frac{X_a - X_b}{fX_a + (1-f)X_b}\right)^2\right] \begin{cases} <0 \text{ if } \lambda < \frac{1}{2}, \\ =0 \text{ if } \lambda = \frac{1}{2}, \\ >0 \text{ if } \lambda > \frac{1}{2}. \end{cases}$$

The first order condition (A.1) is 0 when $f = g = f^{Kelly}$, so when g is near f^{Kelly}, the sign of the first order condition is determined by whether λ is greater than, equal to, or

less than 1/2. For example, if $\lambda < 1/2$, then the derivative of the first order condition (A.1) with respect to g is negative, which implies that the first order condition is negative when $g = f^{Kelly} + \epsilon$, where ϵ is a small positive quantity. Therefore, when $g = f^{Kelly} + \epsilon$, f_1^* is smaller than f^{Kelly}. The cases when $\lambda > 1/2$ and $\lambda = 1/2$ follow similarly. □

Proof of Proposition 5 The first partial derivative of $\mathbb{E}[q_T^f]$ to f is:

$$\frac{\partial \mathbb{E}[q_T^f]}{\partial f} = \frac{1-\lambda}{\lambda}\mathbb{E}\left[\frac{\exp(TR_T^f)\sum_{t=1}^{T}\frac{X_{at}-X_{bt}}{fX_{at}+(1-f)X_{bt}}}{\left(1+\frac{1-\lambda}{\lambda}\exp(TR_T^f)\right)^2}\right].$$

$\mathbb{E}[q_T^f]$ is not necessarily concave, but it is unimodel. The rest follows from similar calculations to Proposition 2. □

Proof of Proposition 6 The first partial derivative of $\mathbb{E}[q_T^f]$ to f evaluated at $f = g$ is given by:

$$\left.\frac{\partial \mathbb{E}[q_T^f]}{\partial f}\right|_{f=g} = T\lambda(1-\lambda)\mathbb{E}\left[\frac{(X_a - X_b)}{fX_a + (1-f)X_b}\right].$$

The cross partial derivative of $\mathbb{E}[q_T^f]$ evaluated at $f = g$ is given by:

$$\left.\frac{\partial^2 \mathbb{E}[q_T^f]}{\partial f \partial g}\right|_{f=g} = 2\lambda(1-\lambda)\left(\lambda - \frac{1}{2}\right)\mathbb{E}\left[\left(\sum_{t=1}^{T}\frac{X_{a,t} - X_{b,t}}{fX_{a,t} + (1-f)X_{b,t}}\right)^2\right].$$

The rest follows similarly to the proof of Propositions 3–4. □

Proof of Proposition 7 It follows directly from (6) and (11).

□

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