On Regularity of Stationary Measures on Weakly Contracting Systems

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Introduction

- Aim: To study (non one-parameter) group/semigroup actions (smooth action) on manifolds.
- Questions:
  - What are orbits like?
  - What are minimal sets like?
  - How to describe dynamics (for some generic elements in the group)?

- A common method is to consider about a random walk induced by the group action.
- As an analogy of invariant measures in the non random context, we care about stationary measures of the random walk.
  (Note that a common invariant measure for the group action in general does not exist.)
- The stationary measures reflect some topological properties:
  - The support is a G-invariant closed subset.
  - Sometimes we can show the support (of an ergodic stationary measure) is a G-minimal set.

For example,

Theorem (Malicet, 2017)

Let $\omega \mapsto (f_\omega^n)$ be a non degenerated random walk on a sub semigroup Γ of $\mathrm{Homeo}(\mathbb{S}^1)$, such that Γ does not preserve a common invariant probability measure on $\mathbb{S}^1$. Then there is only a finite number of ergodic stationary probability measures $\mu_1, \cdots, \mu_p$. Their topological supports $F_1, \cdots, F_p$ are pairwise disjoint and are exactly the minimal invariant compacts of Γ.

Introduction

Another application of the stationary measure is Furstenberg's result of non-commuting random product.

Theorem (Furstenberg, 1963)

Let $A_0, A_1, \cdots, A_n, \cdots$ be a sequence of independent and identically distributed random matrices in $\mathrm{SL}(d, \mathbb{R})$. If the common distribution satisfies some non degenerated conditions (strongly irreducible, non-compact, integrable), then $\exists \lambda > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \log \|A_{n-1} \cdots A_1 A_0\| = \lambda \text{ a.s.}$$

Introduction

$\mathrm{SL}(d, \mathbb{R})$ has a natural linear action on $\mathbb{RP}^{d-1}$. In Furstenberg's proof and all other arguments, they focused on the stationary measure on the projective space.

Later, Guivarc'h showed that the stationary measure is Hölder regular.

Theorem (Guivarc'h, 1990)

Let $A_0, A_1, \cdots, A_n, \cdots$ be a sequence of independent and identically distributed random matrices in $\mathrm{SL}(d, \mathbb{R})$. If the common distribution satisfies some stronger conditions (strong irreducible, proximal, exists exponential moment). Then the unique stationary measure of the induced random walk on $\mathbb{RP}^{d-1}$ is Hölder regular.

Motivation

- To generalize Guivarc'h's theorem to a non-linear case. At least for a smooth perturbation of linear maps on the projective space.

- Guivarc'h's argument highly depends on linear settings: applying Iwasawa decomposition to calculate the action accurately.
- We need to give a new view of smooth actions.

Basic settings

- M is a closed Riemannian manifold.
- $\mathrm{Diff}^{1,\alpha}(M)$ is the family of $C^{1+\alpha}$ diffeomorphisms on M.
- Let μ be a finitely supported probability measure on $\mathrm{Diff}^{1,\alpha}(M)$.
- Let Γ be the semigroup generated by $\operatorname{supp}\mu$.

- Let ν be a Borel probability measure on M, ν is called μ-**stationary** if $\mu * \nu = \nu$, i.e.

$$\int f_*(\nu)\, \mathrm{d}\mu(f) = \nu.$$

- The probability measure ν is called of **Hölder regular** if exists $C, \alpha > 0$, such that for every $x \in M$ and $\varepsilon > 0$,

$$\nu(B(x,\varepsilon)) < C\varepsilon^{\alpha}.$$

Contraction properties

For a random dynamical system, we introduce the weak contraction property.

Definition (weakly contracting)

Let μ be a finitely supported probability measure on $\mathrm{Diff}^{1,\alpha}(M)$. We call μ is weakly contracting, if for any $(x,v) \in TM \setminus \{0\}$, there exists positive integers $N_+ = N_+(x,v)$, $N_- = N_-(x,v)$ such that

$$\frac{1}{N_+}\int \log \frac{\|D_x f(v)\|}{\|v\|} d\mu^{*N_+}(f) < 0, \quad \frac{1}{N_-}\int \log \frac{\|D_x f^{-1}(v)\|}{\|v\|} d\mu^{*N_-}(f) < 0.$$

Here $\mu^{*N} = \mu * \mu \cdots * \mu$ is the N-th convolution power of μ.

Contraction properties

Definition (uniformly contracting)

If there exists $N \in \mathbb{Z}_+$ and $C > 0$, such that for every $(x, v) \in TM \setminus \{0\}$,

$$\frac{1}{N}\int \log \frac{\|D_x f(v)\|}{\|v\|}\, \mathrm{d}\mu^{*N}(f) < -C, \quad \frac{1}{N}\int \log \frac{\|D_x f^{-1}(v)\|}{\|v\|}\, \mathrm{d}\mu^{*N}(f) < -C.$$

Definition (strongly contracting)

If there exists $N \in \mathbb{Z}_+$ and $C > 0$, such that for every $x \in M$,

$$\frac{1}{N}\int \log \|D_x f\|\, \mathrm{d}\mu^{*N}(f) < -C, \quad \frac{1}{N}\int \log \|D_x f^{-1}\|\, \mathrm{d}\mu^{*N}(f) < -C.$$

Contracting and expanding

Proposition

Weak contraction $\iff$ uniform contraction $\iff$ strong contraction.

We compare with a uniform expansion property introduced by Liu-Xu and Chung.

Uniformly expanding

If there exists $N \in \mathbb{Z}_+$ and $C > 0$, such that for every $(x, v) \in TM \setminus \{0\}$,

$$\frac{1}{N}\int \log \frac{\|D_x f(v)\|}{\|v\|} \, \mathrm{d}\mu^{*N}(f) > C, \quad \frac{1}{N}\int \log \frac{\|D_x f^{-1}(v)\|}{\|v\|} \, \mathrm{d}\mu^{*N}(f) > C.$$

The point is: in an expanding context, the uniform expansion does not lead to a "strong expansion".

Main results

Theorem (J.-Xu, in preparation)

If μ is a weakly contracting measure, then every μ-stationary probability measure is Hölder regular.

Corollary

Let $\Gamma \subset \mathrm{Diff}^{1,\alpha}(M)$ be a finitely generated sub semigroup, assume that there is a finite set S supports a weakly contracting measure and $\langle S \rangle = \Gamma$, then every Γ-orbit closure is with positive Hausdorff dimension.

Application: semigroup actions on $\mathrm{Diff}^{1+}(\mathbb{S}^1)$

Theorem (J.-Xu, in preparation)

Let μ be a finitely supported probability measure on $\mathrm{Diff}^{1,\alpha}(\mathbb{S}^1)$, assume that $\operatorname{supp}\mu$ has no common invariant probability measure on $\mathbb{S}^1$. Then every μ-stationary probability measure is Hölder regular.

Corollary

If Γ is a finitely generated sub-semigroup of $\mathrm{Diff}^{1,\alpha}(\mathbb{S}^1)$, then either there exists a Γ-invariant probability measure on $\mathbb{S}^1$, or any Γ-orbit closure has positive Hausdorff dimension.

Remark

If Γ preserves a common invariant probability measure on $\mathbb{S}^1$, then either Γ has a finite orbit or Γ semi-conjugate to a semigroup of rotations on $\mathbb{S}^1$.

Application: perturbation of linear actions on projective spaces

Theorem (J.-Xu, in preparation)

Let $S = \{g_1, \cdots, g_k\} \subset \mathrm{GL}(d, \mathbb{R})$ be a subset such that the semigroup generated by S is Zariski dense in $\mathrm{GL}(d, \mathbb{R})$. Let $\boldsymbol{p} = (p_1, p_2, \cdots, p_k)$ be a non-degenerated probability vector. Then for every $S' = \{f_1, \cdots, f_k\} \subset \mathrm{Diff}^{1,\alpha}(\mathbb{RP}^{d-1})$ which is C^1-closed to S, let

$$\mu = p_1\delta_{f_1} + \cdots + p_k\delta_{f_k}$$

be a probability measure on $\mathrm{Diff}^{1,\alpha}(\mathbb{RP}^{d-1})$, then every μ-stationary probability measure on $\mathbb{RP}^{d-1}$ is Hölder regular.

A same topological corollary holds.

Further discussions

Some ongoing works and further questions:

- Exposit the phenomenon of positive Hausdorff dimension of orbit closures.
- Discuss the uniqueness of stationary measure.
- Whether a μ-stationary measure ν (on $\mathbb{S}^1$) is Rajchman, i.e. the Fourier coefficients $\widehat{\nu}(k) \to 0$.
- For a general 0-entropy system, whether we can get a uniformly contracting system after a perturbation.

Thank you!