The Jacquet-Zagier Trace Formula for $\mathrm{GL}(n)$

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Outline

- Motivations:
  (1). Selberg's trace formula and its generalizations
  (2). Some holomorphy conjectures
- Main results: Expansion of the Jacquet-Zagier trace formula

$$I_0^f(s) = \int_{\mathrm{GL}(n,F)Z(\mathbb{A}_F)\backslash \mathrm{GL}(n,\mathbb{A}_F)} K_0^f(x,x)E(x,s)dx$$

- Describe the trace formula & Idea of proof
- Applications: holomorphy conjectures (Dedekind, Artin, Selberg, ...); nonvanishing
- Further discussions

Introduction

- G: locally compact group, e.g., $G = \mathrm{SL}_2(\mathbb{R})$
- Γ: discrete subgroup of G with $\mathrm{Vol}(\Gamma\backslash G) < \infty$, e.g., $\Gamma = \mathrm{SL}_2(\mathbb{Z})$

Problem:

Study the spectral decomposition of $L^2(\Gamma\backslash G)$. (discrete & continuous)

- Selberg studied Hecke operators (from harmonic analysis point of view), for $f \in C_c(\mathrm{SL}_2(\mathbb{R})//\mathrm{SO}_2(\mathbb{R}))$, the Hecke operator

$$\phi \xrightarrow{\mathrm{R(f)}} f * \phi : \ g \mapsto \int_{\mathrm{SL}_2(\mathbb{R})} \phi(gx) f(x) dx.$$

- characterize the continuous spectrum [Eisenstein series]
- compute the trace of $R(f)$ on the discrete spectrum (space of Maass forms) as a sum of orbital integrals [Selberg's Trace Formula]

Selberg's Trace Formula

- Integral representation:

$$(R(f)\phi)(x) = \int_{\Gamma\backslash G} \mathrm{K}(x,y)\phi(y)dy, \quad \mathrm{K}(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y).$$

- Compute the trace:

$$\mathrm{Tr}\, R(f) = \int_{\Gamma\backslash G} \mathrm{K}(x,x)dx = \sum_{\{\gamma\}} \mathrm{Vol}(\Gamma_\gamma\backslash G_\gamma) \underbrace{\int_{G_\gamma\backslash G} f(x^{-1}\gamma x)dx}_{\triangleq\ \mathrm{Orb}(\gamma)\ \text{(orbital integrals)}}.$$

Generalization:

- *study of Eisenstein series and characterize continuous spectrum [Langlands, 1976]*
- *$R(f)$ must be of trace class on the discrete spectrum [Müller, 1989]*
- *generalize the trace formula to reductive groups [Arthur, since 1974]*

The Arthur-Selberg Trace Formula

- Trace formula

$$\underbrace{\sum_{\pi} \operatorname{Tr} \pi(f) + \boxed{\text{cont. spec.}}}_{\text{the spectral side}} = \operatorname{Tr} R(f) = \underbrace{\sum_{\{\gamma\}} \operatorname{Vol}(\Gamma_\gamma \backslash G_\gamma) \operatorname{Orb}(\gamma)}_{\text{the geometric side}} \quad (1)$$

- In general, the formula (1) does not converge. So a suitable truncation is needed to make it convergent.
- Arthur's truncation: geometric truncation $\mathrm{K}^T(x, y)$ and spectral truncation $\Lambda_2^T \mathrm{K}(x, y)$, moreover, when substituted into (1), the equality still holds and is well defined.

The Arthur-Selberg Trace Formula

- Geometric truncation $\mathrm{K}^T(x,x)$: obtained by certain combinatorics and reduction theory, such that $\mathrm{K}^T(x,x) \to \mathrm{K}(x,x)$ as $T \to \infty$ and

$$\int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \mathrm{K}^T(x,x)dx = \text{a polynomial in } T.$$

- Spectral truncation $\Lambda_2^T \mathrm{K}(x,y)$: by Langlands theory on Eisenstein series, $\mathrm{K}(x,y)$ is equal to

$$\sum_{\chi\in\mathfrak{X}} \sum_{P\in\mathcal{P}} \frac{1}{k_P!(2\pi)^{k_P}} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \sum_{\phi\in\mathfrak{B}_{P,\chi}} E(x, \mathcal{I}_P(\lambda,f)\phi,\lambda)\overline{E(y,\phi,\lambda)}d\lambda.$$

Λ^T is a truncation operator and $\Lambda_2^T \mathrm{K}(x,y)$ denotes the operator acts on the second Eisenstein series.

$$\int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \Lambda_2^T \mathrm{K}(x,x)dx = \int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \mathrm{K}^T(x,x)dx.$$

Jacquet–Zagier's Approach to Trace Formulas on GL(2)

Motivations:

- *the truncation process on the* $\mathrm{K}(x,x)$ *is somewhat too complicated to lead to explicit forms in many situations*
- *Jacquet and Zagier initiated a new approach by introducing the Rankin–Selberg method into the treatment of* $\mathrm{K}(x,x)$ *for* $\mathrm{GL}(2)$.
- *Main goals:*
  - (1). *derive the Selberg trace formula, avoiding the recourse to Arthur's truncation*
  - (2). *prove holomorphic continuation of symmetric square L-functions for* $\mathrm{GL}(2)$
  - (3). *connections between adjoint L-functions (analytic) and certain Artin L-functions (algebraic)*

Jacquet–Zagier's Trace Formula

- $E(x,s)$: Eisenstein series with $\operatorname{Res}_{s=1} E(x,s) = 1$
- Consider the function

$$I(s) = \int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0(x,x)E(x,s)dx.$$

- Compute the geometric side and the spectral side as a meromorphic function of s
- **Conjecture:** $I(s)/\zeta_F(s)$ **is entire!** [Holomorphy of $L(s,\pi,\mathrm{Sym}^2)$]
- Taking residue at $s=1$ then $\operatorname{Res}_{s=1} I(s) = \operatorname{Tr} R(f)$. Moreover,

$$\underbrace{\sum_{\pi} \operatorname{Res}_{s=1} L(s,\pi\times\widetilde{\pi}) + \cdots}_{\text{the spectral side}} = \operatorname{Tr} R(f) = \underbrace{\sum_{E} \operatorname{Res}_{s=1} \zeta_E(s) + \cdots}_{\text{the geometric side}}$$

Our Goal: generalization to higher ranks

Goal:

- *$G = \mathrm{GL}(n)$ over a global field F*
- *Consider the function*

$$I(s) = \int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \mathrm{K}(x, x)E(x, s)dx.$$

- *Compute the geometric and the spectral sides as meromorphic functions of s*
- *When $n \leq 4$, we prove $I(s)/\zeta_F(s)$ is entire and deduce the holomorphy of $L(s, \pi, \mathrm{Ad})$ [Selberg's conjecture]*

Framework of trace formulas ($G = \mathrm{GL}(n)$)

- $f : G(\mathbb{A}_F) \to \mathbb{C}$, left and right K-finite, compact supp mod $Z_G(\mathbb{A}_F)$
- f defines an integral operator

$$R(f)\phi(y) = \int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} f(x)\phi(yx)dx,$$

on the space $L^2(G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F))$ with the kernel function

$$K^f(x,y) = \sum_{\gamma \in Z_G(F)\backslash G(F)} f(x^{-1}\gamma y).$$

- Spectral decomposition:

$$K^f(x,y) = K_0^f(x,y) + K_{\mathrm{ER}}^f(x,y)$$

- Arthur-Selberg trace formula computes expansions of

$$\mathrm{Tr}\, R_0(f) = \int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} K_0^f(x,x)dx = \text{Geometric} - \text{Non-cuspidal}$$

Notation

- τ : Hecke character; Φ : Bruhat-Schwartz function on $\mathbb{A}_F^n$

$$h_\tau(x,s) = \tau(\det x)|\det x|^s \int_{\mathbb{A}_F^\times} \Phi((0,\cdots,0,t)x)\tau(t)|t|^{ns} d^\times t$$

- Eisenstein series:

$$E_\tau(x,s) = \sum_{\gamma \in P(F)\backslash G(F)} h_\tau(\gamma x, s), \ \Re(s) > 1,$$

with meromorphic continuation and F.E.; moreover,

$$\operatorname*{Res}_{s=1} E_\tau(x,s) = \frac{\widehat{\Phi}(0)\tau(\det x)}{2}$$

On $\mathrm{GL}(n)$: Decomposition of $K^f(x,x)$

- Define the distribution

$$I_0^f(s,\tau) = \int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} K_0^f(x,x)E_\tau(x,s)dx$$

- Spectral expansion

$$I_0^f(s,\tau) \sim \sum_{\pi} L(s, \pi \times \widetilde{\pi} \otimes \tau)$$

- Selberg conjecture is equivalent to the holomorphy of $I_0^f(s,\tau)/\Lambda(s,\tau)$

Goal:

Study the expansion of $I_0^f(s,\tau)$ in terms of its geometric and non-cuspidal side.

On $\mathrm{GL}(n)$: Decomposition of $K^f(x,x)$

- Trivially, $I_0^f(s,\tau) = I^f(s,\tau) - I_{\mathrm{ER}}^f(s,\tau)$

$$I_*^f(s,\tau) = \int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \mathrm{K}_*^f(x,x)\cdot E_\tau(x,s)dx,\ * \in \{\emptyset, er\}$$

- $I^f(s,\tau)$ and $I_{\mathrm{ER}}^f(s,\tau)$ do not converge
- Truncation does not fit well
- Poisson summation to $E_\tau(x,s)$ will lose L-function structure
- Need new ideas to regularize them. Starting with

$$I^f(s,\tau) = \int_{Z_G(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in Z_G(F)\backslash G(F)} f(x^{-1}\gamma x)\cdot h_\tau(x,s)dx$$

Decomposition of $K^f(x,x)$

- P: standard parabolic subgroup of type $(n-1,1)$
- $\mathcal{Q}$: set of standard parabolic subgroups
- $\mathfrak{S} := \{p^{-1}\gamma p : \ p \in P(F), \ \gamma \in Z_G(F)\backslash Q(F), \ Q \in \mathcal{Q}\}$
- Define

$$\mathrm{K}_{\text{geo,reg}}(x,y) = \sum_{\gamma \in Z_G(F)\backslash G(F) - \mathfrak{S}} f(x^{-1}\gamma y)$$

$$\mathrm{K}_{\text{geo,sing}}(x,y) = \sum_{\gamma \in \mathfrak{S}} f(x^{-1}\gamma y)$$

- Can show

$$Z_G(F)\backslash G(F) - \mathfrak{S} = \bigsqcup \text{regular } P(F)\text{-conjugacy classes}$$

Decomposition of $K_{\mathrm{ER}}^f(x,x) = K_{\mathrm{Eis}}^f(x,x) + \mathrm{K}_{\mathrm{Res}}^f(x,x)$

- $K_{\mathrm{ER}}^f(x,y)$ is equal to

$$\sum_{\chi\in\mathfrak{X}^{\mathrm{prop}}}\sum_{P\in\mathcal{P}}\frac{1}{k_P!(2\pi)^{k_P}}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\sum_{\phi\in\mathfrak{B}_{P,\chi}} E(x,\mathcal{I}_P(\lambda,f)\phi,\lambda)\overline{E(y,\phi,\lambda)}d\lambda.$$

- apply Fourier expansion to Eisenstein series $E(x,\mathcal{I}_P(\lambda,f)\phi,\lambda)$:

$$K_{\mathrm{ER}}^f(x,y) = \int_{[N_P]}\mathrm{K}(ux,x)du + \sum_{i=2}^{n-1}\underbrace{\mathcal{F}_1^{(i)}\mathrm{K}(x,x)}_{\text{partial Fourier transform of K}} + \underbrace{\mathrm{K}_{\mathrm{Whi}}(x,x)}_{\text{generic part}}$$

- Explicitly,

$$\mathrm{K}_{\mathrm{Whi}}(x,x) = \sum_{\delta\in N(F)\backslash P_0(F)}\int_{[N]}\mathrm{K}_{\mathrm{Eis}}(u\delta x,x)\theta(u)du,$$

Decomposition of $K_0^f(x,x)$

- Define

$$\mathrm{K}_{P,\mathrm{reg}}(x,x) = \int_{[N_P]} \mathrm{K}_{\mathrm{geo,reg}}(ux,x)du$$

$$\mathrm{K}_{\mathrm{Whi}}(x,x) = \sum_{\delta\in N(F)\backslash P_0(F)} \int_{[N]} \mathrm{K}_{\mathrm{Eis}}(u\delta x,x)\theta(u)du$$

$$\mathrm{K}_{\mathrm{sing}}(x,x) = \mathrm{K}_{\mathrm{geo,sing}}(x,x) - \int_{[N_P]} \mathrm{K}_{\mathrm{geo,sing}}(ux,x)du - \sum_{i=2}^{n-1} \mathcal{F}_1^{(i)}\mathrm{K}(x,x)$$

- Then we have

$$K_0^f(x,x) = \underbrace{K^f_{\mathrm{geo,reg}}(x,x) - K^f_{P,\mathrm{reg}}(x,x) + K^f_{\mathrm{sing}}(x,x)}_{\text{geometric}} - \underbrace{K^f_{\mathrm{Whi}}(x,x)}_{\textit{spectral}}$$

Decomposition of $I_0^f(s,\tau)$

$$I^f_{\text{geo,reg}}(s,\tau) := \int_{P(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_{\text{geo,reg}}(x,x)h_\tau(x,s)dx,$$

$$I^f_{P,\text{reg}}(s,\tau) := \int_{P(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \int_{[N_P]} \mathrm{K}_{\text{geo,reg}}(ux,x)h_\tau(x,s)dudx$$

$$I^f_{\text{Whi}}(s,\tau) := \int_{N(\mathbb{A}_F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \int_{[N]} \int_{[N]} \mathrm{K}_{\text{Eis}}(ux,vx)\theta(u)\overline{\theta}(v)h_\tau(x,s)dudvdx$$

$$I^f_{\text{sing}}(s,\tau) := \int_{P(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_{\text{sing}}(x,x)h_\tau(x,s)dx,$$

- Starting Point:

$$I_0^f(s,\tau) = I^f_{\text{geo,reg}}(s,\tau) - I^f_{P,\text{reg}}(s,\tau) + I^f_{\text{sing}}(s,\tau) - I^f_{\text{Whi}}(s,\tau)$$

- Will compute each integral (in terms of various L-functions)

Jacquet-Zagier Trace Formula for $\mathrm{GL}(n)$

Very roughly, we have

Theorem

Let $\Re(s) > 1$. Then we have

$$\underbrace{I_0^f(s,\tau)}_{\substack{\text{infinite sum of}\\ L(s,\pi\otimes\tau\times\widetilde{\pi})}} = \underbrace{I_{\text{geo,reg}}^f(s,\tau)}_{\text{finite sum of Hecke } L\text{-functions}} - \underbrace{I_{P,\text{reg}}^f(s,\tau)}_{\substack{\text{Langlands-Shahidi}\\ \&\text{Godement-Jacquet}}}$$

$$- \underbrace{I_{\text{Whi}}^f(s,\tau)}_{\substack{\text{Rankin-Selberg}\\ \text{for non-cuspidal}}} + \underbrace{I_{\text{sing}}^f(s,\tau)}_{\substack{\text{various } L\text{-functions}\\ \text{on smaller groups/ranks}}}$$

- Meromorphic continuation to $\mathbb{C}$
- When $\tau^k \neq 1$ for $1 \leq k \leq n$, get *analytic* continuation to $\mathbb{C}$

$I_{\mathrm{geo,reg}}(s,\tau)$: Hecke L-series attached to Étale algebras

- Recall

$$I^f_{\mathrm{geo,reg}}(s,\tau) := \int_{P(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{\gamma\in Z_G(G)\backslash G(F)-\mathfrak{S}} f(x^{-1}\gamma x)h_\tau(x,s)dx$$

Upshot:

$$I^f_{\mathrm{geo,reg}}(s,\tau) = \sum_{g=1}^{n} \sum_{\substack{\mathbf{f},\mathbf{e}\in\mathbb{N}^g \\ \langle\mathbf{f},\mathbf{e}\rangle=n}} C_{\mathbf{f},\mathbf{e}} \prod_{i=1}^{g} \frac{1}{f_i^{e_i}} \sum_{[E_i:F]=f_i} Q_{E_i}(s) \prod_{j=1}^{e_i} L[j]\left(s,\tau\circ N_{E_i/F}\right),$$

where $L[j](s,\chi) := L(js-j+1,\chi^j)$.

- hyperbolic: $L(s,\tau)^2$; uniponent: $L(s,\tau)L(2s-1,\tau^2)$
- elliptic regular: $L(s,\tau\circ N_{E/F})$, $[E:F]=2$

$I_{P,\mathrm{reg}}(s,\tau)$: Intertwining operators

- Recall

$$I^{f}_{P,\mathrm{reg}}(s,\tau) := \int_{P(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \int_{[N_P]} \mathrm{K}_{\mathrm{geo,reg}}(ux,x) h_\tau(x,s)\, du\, dx$$

- Explicit representatives for $Z_G(F)\backslash G(F) - \mathfrak{S}$ by Ad $P(F)$:

$$\left\{ w_1 w_2 \cdots w_{n-1} \begin{pmatrix} I_{n-3} & & \\ & t & \\ & & I_2 \end{pmatrix} \mathfrak{u} : \ t \in F^\times / (F^\times)^n,\ \mathfrak{u} \in N_P(F) \right\}$$

Upshot:

$$I_{P,\mathrm{reg}}(s,\tau) = Q(s) \cdot \frac{L(s,\tau)L(2s,\tau^2)\cdots L((n-1)s,\tau^{n-1}) L(ns,\tau^n)}{L(s+1,\tau)L(2s+1,\tau^2)\cdots L((n-1)s+1,\tau^{n-1})}$$

$I_{\mathrm{Whi}}(s,\tau)$: Rankin-Selberg periods for non-cuspidal reps

$$I_{\mathrm{Whi}}(s,\tau)=\sum_{\chi}\sum_{P\in\mathcal{P}}\frac{1}{c_P}\sum_{\substack{\phi_1\in\mathfrak{B}_{P,\chi}\\ \phi_2\in\mathfrak{B}_{P,\chi}}}\int_{(i\mathbb{R})^{r_P-1}}\langle\mathcal{I}_P(\lambda,f)\phi_2,\phi_1\rangle\mathcal{P}(\phi_1,\overline{\phi}_2;\lambda)d\lambda,$$

where $\mathcal{P}(\phi_1,\overline{\phi}_2;\lambda)$ is the Rankin-Selberg period:

$$\mathcal{P}(\phi_1,\overline{\phi}_2;\lambda)=\int_{Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}W_1(x;\lambda)\overline{W_2(x;\lambda)}h_\tau(x,s)dx.$$

- infinite sum over $\chi\in\mathfrak{X}^{\mathrm{prop}}$, absolutely convergent in $\Re(s)>1$ (reduce it to a RTF which can be majorized by gauges)
- meromorphic continuation

Meromorphic Continuation

- $\phi_i \in \pi_\lambda = \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)} \left(\pi_1 \otimes |\cdot|_{\mathbb{A}_F}^{\lambda_1}, \pi_2 \otimes |\cdot|_{\mathbb{A}_F}^{\lambda_2}, \cdots, \pi_r \otimes |\cdot|_{\mathbb{A}_F}^{\lambda_r} \right)$
- the quotient

$$\frac{\mathcal{P}(\phi_1, \overline{\phi}_2; \lambda)}{L(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})} = \text{a finite product of entire functions}$$

- Here

$$L(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda}) = \prod_{i=1}^{n} \prod_{j=1}^{n} L(s + \lambda_i - \lambda_j, \pi_i \otimes \tau \times \widetilde{\pi}_j)$$

- obtain meromorphic continuation of $\mathcal{P}(\phi_1, \overline{\phi}_2; \lambda)$
- continuation of $I_{\mathrm{Whi}}(s, \tau)$: shift contour of the integral w.r.t. the multiple complex variable λ

$I_{\text{sing}}(s, \tau)$: Singular terms

Upshot:

$I_{\text{sing}}(s, \tau)$ is holomorphic in the right half plane $\Re(s) \geq 1/2$ if $s \notin \{1/2, 1\}$; and it may have at most simple pole at $s = 1/2$ if $\tau^2 = \mathbf{1}$.

- Recall

$$I_{\text{sing}}(s, \tau) := \int_{Z(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)} \underbrace{K_{\text{sing}}(x, x)}_{n \geq 3} h_\tau(x, s) dx.$$

- Bruhat decomposition; singular parts of $K_{\text{sing}}(x, x)$ disappear
- Can be written as a finite linear combination of orbital integrals from smaller groups (i.e., Levi of G)
- Convergence follows from induction if $\Re(s) > 1$
- Meromorphic continuation to $\Re(s) \geq 1/2$ and apply the F.E.

Jacquet-Zagier Trace Formula for $\mathrm{GL}(n)$

Theorem

$$\underbrace{I_0^f(s,\tau)}_{L(s,\pi\otimes\tau\times\widetilde{\pi})} = \underbrace{I_{\text{geo,reg}}^f(s,\tau)}_{\textit{Hecke}} - \underbrace{I_{P,\text{reg}}^f(s,\tau)}_{\substack{\textit{Langlands-Shahidi}\\ \textit{\& Godement-Jacquet}}} - \underbrace{I_{\text{Whi}}^f(s,\tau)}_{\substack{\textit{Rankin-Selberg}\\ \textit{for non-cuspidal}}} + \underbrace{I_{\text{sing}}^f(s,\tau)}_{\substack{\textit{L-functions}\\ \textit{on smaller ranks}}}$$

- Next Goal: investigate analytic behaviors of $I_0^f(s,\tau)/L(s,\tau)$
- Show $\frac{I_{P,\text{reg}}^f(s,\tau)}{L(s,\tau)}$, $\frac{I_{\text{Whi}}^f(s,\tau)}{L(s,\tau)}$, and $\frac{I_{\text{sing}}^f(s,\tau)}{L(s,\tau)}$ are holomorphic
- As a consequence,

$$\underbrace{L(s,\pi,\mathrm{Ad}\otimes\tau)}_{\text{analytic}} \approx \frac{I_0^f(s,\tau)}{L(s,\tau)} \sim \frac{I_{\text{geo,reg}}^f(s,\tau)}{L(s,\tau)} \approx \underbrace{\frac{L(s,\tau\circ N_{E/F})}{L(s,\tau)}}_{\text{algebraic}}$$

Main Application

- F : global field
- τ : idele class character on $\mathbb{A}_F^\times$
- π : a cuspidal representation on $GL(n, \mathbb{A}_F)$
- $L(s, \pi, \mathrm{Ad} \otimes \tau) := L(s, \pi \times \widetilde{\pi} \otimes \tau)/L(s, \tau)$

Theorem

Let $n \leq 4$. Then the **complete** *L-function $L(s, \pi, \mathrm{Ad} \otimes \tau)$ is entire, unless $\tau \neq 1$ and $\pi \otimes \tau \simeq \pi$, in which case $L(s, \pi, \mathrm{Ad} \otimes \tau)$ is meromorphic with only simple poles at $s = 0, 1$.*

- Potential application in converse theorem
- $n = 2$: Gelbart-Jacquet lifting

The Jacquet-Zagier Trace Formula

Artin Conjecture $\Longleftarrow$ Langlands Philosophy

Dedekind Conjecture: $\zeta_E(s)/\zeta_F(s)$ is entire

holds for **degree** $n = [E : F] \leq 4$

$I^f_{\text{geo,reg}}(s,\tau)/L(s,\tau)$

J-Z trace formula $\longrightarrow$

Selberg Conjecture: $L(s,\pi,\text{Ad})$ entire

$\pi \in \mathcal{A}_0(GL(n))$, **rank** $n \leq 4$

$I^f_0(s,\tau)/L(s,\tau)$

The Reverse Direction (easier)

Theorem

Let notation be as before. Assume the adjoint L-functions $L(s,\pi,\mathrm{Ad})$ are holomorphic for all π with a supercuspidal component at one place v. Then the Dedekind conjecture holds for all field extensions of E/F of degree n.

- Idea of proof:

$$\underbrace{\text{Dedekind Conjecture: } \zeta_E(s)/\zeta_F(s) \text{ is entire}}_{\text{holds for } \textbf{degree } n=[E:F]} \xleftarrow{\text{J-Z trace formula}} \underbrace{L(s,\pi,\mathrm{Ad}) \text{ entire}}_{\pi\in\mathcal{A}_0(GL(n)),\ \textbf{rank } n}$$

$$I^f_{\mathrm{geo,reg}}(s,\tau)/L(s,\tau) \qquad\qquad I^f_0(s,\tau)/L(s,\tau)$$

- Holds for all $n \geq 2$

Nonvanishing of Central L-values

Theorem

Let $n \geq 2$. Suppose there exists an extension E/F with degree $[E : F] = n$, and $\zeta_E(1/2) \neq 0$. Then there exists a $\pi = \pi(E) \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F), \omega^{-1})$, such that $L(1/2, \pi \times \widetilde{\pi}) \neq 0$.

- Fröhlich: there are infinitely many number fields F such that

$$\zeta_F(1/2) = 0$$

- Selberg conjecture implies

$$L(1/2, \pi \times \widetilde{\pi}) = 0, \quad \forall\, \pi \in \mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F), \omega^{-1})$$

Further Discussion

- Using the test function of B-P-L-Z-Z, the J-Z TF becomes

$$I_0(s,\tau) = \sum_{\phi\in\mathfrak{B}_\pi} \mathcal{P}(s,\phi,\tau) = \mathrm{Geo}(s,\tau),$$

with

$$\mathcal{P}(s,\phi,\tau) = L(s,\pi\otimes\tau\times\widetilde{\pi})\cdot\prod_v \mathcal{P}_v^\sharp(s,\tau)$$

- Goal: show $L(s,\pi\otimes\tau\times\widetilde{\pi})/L(s,\tau)$ is entire
- Entireness of $\mathcal{P}(s,\phi,\tau)/L(s,\tau)$ is insufficient
- In general,

$$L(s,\pi\otimes\tau\times\widetilde{\pi}) = \sum_\phi \mathcal{P}(s,\phi,\tau)$$

- RTF with an automorphic weight under

$$K_0^f(x,x) = \underbrace{K^f_{\mathrm{geo,reg}}(x,x) - K^f_{P,\mathrm{reg}}(x,x) + K^f_{\mathrm{sing}}(x,x)}_{\text{geometric}} - \underbrace{K^f_{\mathrm{Whi}}(x,x)}_{\textit{spectral}}$$

Thank You!