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姓名：徐恺

学号：1600010715

专业：数学与应用数学

导师：刘毅

四维几何中的支配关系

徐恺

摘要：本文研究四维几何之间的虚拟支配关系。这一关系在四维几何中构成偏序，并可由一个支配图表示。作者作出并证明了支配图。此外，几何X支配几何Y当且仅当$X \times \mathbf{E}^n$支配$Y \times \mathbf{E}^n$。

关键词：几何，四维几何，支配，(李群的)格

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Domination Relations in Four-dimensional Geometries

Kai Xu

Abstract: This article studies the virtual domination relation in four-dimensional geometries. This relation forms a partial order among the geometries, and can be shown in a domination diagram. The author draws and proves the diagram. Furthermore, a geometry X dominates another geometry Y if and only if $X \times \mathbb{E}^n$ dominates $Y \times \mathbb{E}^n$.

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1 Introduction

Assupmtion: All manifolds in this article are assumed to be closed and oriented, unless otherwise indicated.

The beginning of everything is the notion of domination:

Definition 1.1: Given two manifolds M_1 and M_2 of the same dimension, we say that M_1 **dominates** M_2 if there exists a map of non-zero degree from M_1 to M_2.

Domination between manifolds has been extensively studied. [1], for example, gives a topological criterion concerning the domination between highly-connected even-dimensional manifolds. In dimension three, domination problems are in a much more active discussion. In the survey paper [18], Shicheng Wang summarized important problems on the domination between 3-manifolds. Geometric decomposition of 3-manifolds plays an important role, so do geometric manifolds. [13], [19] and [5], for example, studied non-zero degree maps between geometric 3-manifolds.

This article is concentrated on the domination between geometric 4-manifolds. Our target is the following domination relation between geometries:

Definition 1.2: Given two geometries X and Y of the same dimension. We say that X **virtually dominates** Y (abbreviated as: **dominates**), if for every X-manifold M there is a finite cover $M' \to M$ such that M' dominates some Y-manifold N. The domination relation is written as $X \to Y$.

It is easy to see that the domination relation is transitive: Let X, Y, Z be three geometries. if $X \to Y$ and $Y \to Z$, then $X \to Z$.

The domination relation between four-dimensional geometries is shown in the following diagram:

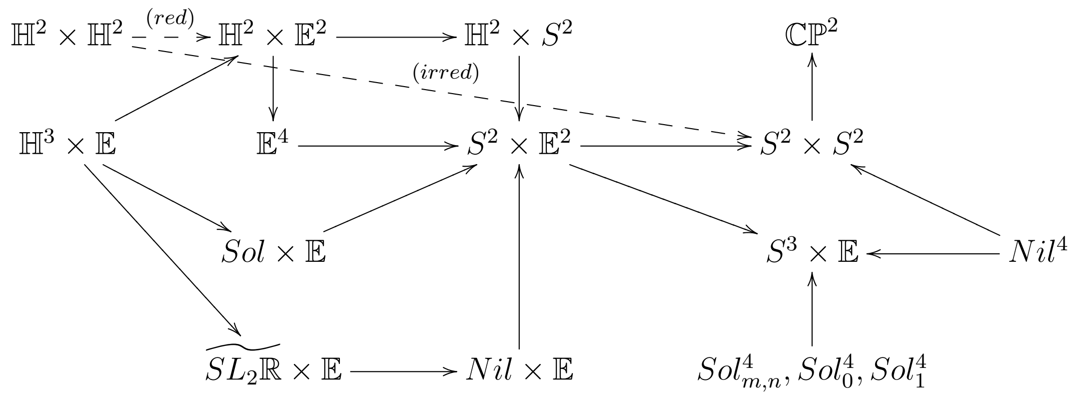


Diagram 1-1

Note that the arrow from $\mathbb{H}^2 \times \mathbb{H}^2$ separates into two parts: **reducible** ("red" in the diagram) and **irreducible** ("irred" in the diagram). Reducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds are finitely coverd by a product of surfaces, whilc irreducible manifolds are not. This notion of irreducibility is essentially the same as the irreducibility of lattices in semisimple Lie groups.

The behavior of reducible and irreducible manifolds in domination are quite different. As we can see from the diagram, reducible manifolds (virtually) dominates $\mathbb{H}^2 \times \mathbb{E}^2$, but irreducible ones can only dominate $S^2 \times S^2$. What happens to irreducible manifolds is closely related to Margulis's Normal Subgroup Theorem; see Section 9 for a detailed explanation.

There are 4 geometries that do not appear in Diagram 1-1: $\mathbb{H}^4$, $\mathbb{CH}^2, F_4$ and S^4. All geometries dominate S^4 since all manifolds dominate S^4. F_4 admits no compact quotient. We do not discuss $\mathbb{H}^4$ and $\mathbb{CH}^2$ in this passage; we know little about them so far.

The domination diagram in dimension three is shown below. It is a part of the proof of Diagram 1-1.

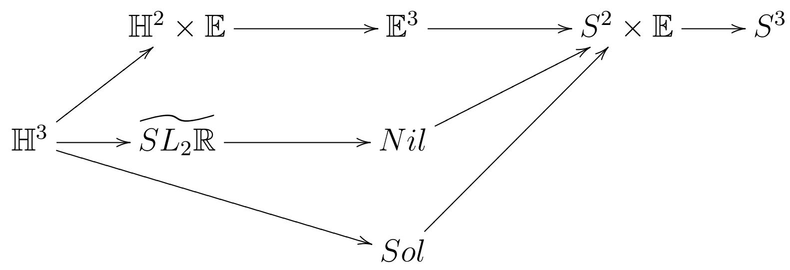


Diagram 1-2

Since there is geometric decomposition in dimension 3 as well as abundant topological tools, people are able to prove much stronger results for 3-manifolds. For example, we can prove that every 3-manifold 1-dominates only finitely many geometric 3-manifolds ([19, Corollary 1]). For another example, we can determine how many numbers of degrees (finitely or infinitely many) could there be in the maps between two 3-manifolds, according to their pieces in the geometric decomposition ([18, Theorem 1.3]).

However, there is no geometric decomposition for 4-manifolds, and there are not as much tools as in dimension three. It is generally hard for us to tell precisely whether a given 4-manifold dominates another one. But if we concentrate on geometric manifolds and virtual domination, we have good and rather complete results.

Insight into the domination diagram can also suggest us of interesting discoveries. Staring at Diagram 1-1 and 1-2, we can find that the map $X \mapsto X \times \mathbb{E}$ embeds Diagram 1-2 into Diagram 1-1 as a derived subgraph. Surprisingly, this holds in higher dimensions:

Corollary 7.5 (Embedding of Domination Diagram) *Suppose X, Y are two geometries of dimension (both) 2, 3 or 4. In addition, $X \neq \mathbb{H}^2 \times \mathbb{H}^2$ when Y is not contractible. Then $X \to Y$ if and only if $X \times \mathbb{E}^n \to Y \times \mathbb{E}^n$.*

This conclusion is mainly based on two theorems:

Proposition 7.1 (Product Geometry Splitting) *Let X be any geometry of dimension 2 or 3 or 4, n be any positive integer. Then every $X \times \mathbb{E}^n$-manifold is finitely covered by $N \times T^n$ where N is an X-manifold.*

Proposition 7.3 (Domination Reduction) *Suppose M_1, M_2 are two manifolds of the same dimension. M_2 is a $K(G,1)$ and $\pi_1(M_2)$ is torsion-free. If $M_1 \times S^1$ dominates $M_2 \times S^1$, then some finite cover of M_1 dominates M_2.*

Proposition 7.1 is proved by investigating the geometries one by one. Proposition 7.3 is purely topological. We still do not know whether Proposition 7.1 generailzes to dimension > 4, since its proof relies hevily on the properties of individual geometries.

Let us return to the proof of Diagram 1-1. In studying virtual domination, it is natural to develop the notion of Typical Covering Manifolds. The typical covering manifolds of a geometry X is a "supreme set" of X-manifolds under covering. That is, every X-manifold is covered by one of them. Studying the domination between between geometries is simplified to studying the domination between typical covering manifolds. Our two main criterions are direct consequence of definitions:

Proposition 5.1 *Let X and Y be geometries. If every typical covering manifold of X dominates some Y-manifold, then $X \to Y$ holds.*

Proposition 6.2 *Assume that X, Y are two geometries, and $\mathcal{A}$ (resp. $\mathcal{B}$) is a set of typical covering manifold for X (resp. Y). If for every $M_1 \in \mathcal{A}$ and $M_2 \in \mathcal{B}$, M_1 does not dominate M_2, then $X \nrightarrow Y$.*

Finding the typical covering manifolds is thus a very important work. Fortunately, for almost all geometries we can find typical covering manifolds that are simple enough. For example, products of lower-dimensional manifolds. Table 3-1 contains our results. Most of the items in Table 3-1 are special cases of Proposition 7.1, and their proofs also generalizes to the proof of Proposition 7.1.

In proving Table 3-1, we frequently use the theorems on the discrete subgroup of Lie groups. By intersecting a lattice with appropriate closed subgroups, we may find sublattices and quotient lattices, and therefore construct fiber bundle structures. The bundle structure becomes simple enough (for example, becomes trivial) after taking suitable covers. This is a commonly used method in studying geometric manifolds. [15] and [16], for example, gives a detailed study of the fibration structures that occur in 4-dimensional geometries.

The remaining non-product geometries are $Nil^4, Sol^4_{m,n}, Sol^4_0$ and Sol^4_1. Their geometric manifolds are mapping torus of T^3 or Nil-manifold. This property is sufficient for us to determine their domination relation with other geometries.

The arrows in Diagram 1-1 will be easy conclusions of section 3 and 4. Proving non-arrows, however, needs more efforts. Some non-arrows are consequences of Diagram 1-2 and Corollary 7.5. Others have to be proved independently, mainly by using $K(G, 1)$. (As we know, $K(G, 1)$ is a bridge between topological information and group-theoretic information.)

The structure of this article is as follows. In section 2 we briefly introduce some basic notions, including the notion of geometry, Seifert bundles, monodromy and Euler numbers of torus bundles, and some discussion on Diagram 1-2. In section 3 we concern about Typical Covering Manifolds, and prove the important Table 3-1. In section 4 we study $Nil^4, Sol^4_{m,n}, Sol^4_0$ and Sol^4_1-manifolds. In section 5, 6 and 8, we complete the proof of Diagram 1-1, except for $\mathbb{H}^2 \times \mathbb{H}^2$. In section 7, we prove the Embedding Theorem for domination relation. Finally in section 9, we discuss $\mathbb{H}^2 \times \mathbb{H}^2$ and complete the whole proof.

2 Preliminaries

1. Four-dimensional Geometries

A **geometry** is a pair (X, G, ρ), where X is a connected simply-connected manifold, G is a **connected** Lie group, and ρ is an effective, transitive left action of G on X with compact point stabilizer. We require two additional conditions:

① There exists a discrete subgroup Γ of G such that $\Gamma\backslash X$ is of finite volume.

② G is required to be maximal among those pairs (X,G',ρ') which satisfies all the conditions above.

We normally abbreviate a geometry (X, G, ρ) to X, and G is called the structure group of X. A manifold M is called to have the geometric structure of X, or to be of type X, or to be an X-manifold, if there is a discrete subgroup $\Gamma \subset G$ acting freely on X such that $M \cong \Gamma\backslash X$. Since X is simply-connected, we have $\pi_1(M) \cong \Gamma$.

Important thing: We require here that the structure group of a geometry be connected. For example, the structure group of Sol is $Iso^0(Sol) = Sol$, which is a subgroup of index 8 in $Iso(Sol)$. In our sense, Sol-manifolds are precisely torus bundle over the circle with Anosov glueing matrix (rather than being covered by them). Our requirement does not lose essential information, since the virtual domination allows passing to covering.

There are 8 three-dimensional geometries: $S^3, \mathbb{E}^3, \mathbb{H}^3, S^2 \times \mathbb{E}, \mathbb{H}^2 \times \mathbb{E}, Nil, \widetilde{SL_2\mathbb{R}}$ and Sol. [12] gives a very good introduction to them, and we assume that the reader is familiar with their properties. The geometric structure on a 3-manifold is unique (if there is one), and according to geometric decomposition, every prime manifold can be cut by tori into geometric manifolds.

Four-dimensional geometries are classified by Filipkiewicz in [2]. The complete list is: $\mathbb{E}^4, S^4, \mathbb{H}^4, \mathbb{CP}^2, \mathbb{CH}^2, S^3 \times \mathbb{E}, \mathbb{H}^3 \times \mathbb{E}, Nil \times \mathbb{E}, Sol \times \mathbb{E}, \widetilde{SL_2\mathbb{R}} \times \mathbb{E}, S^2 \times S^2, \mathbb{E}^2 \times S^2, \mathbb{H}^2 \times S^2, \mathbb{H}^2 \times \mathbb{E}^2, \mathbb{H}^2 \times \mathbb{H}^2, Sol_0^4, F^4, Nil^4, Sol_{m,n}^4, Sol_1^4$. Their precise definitions can be found in the end of Filipkiewicz's article [2, "Summary of Maximal Geometries"].

It is natural to ask about the uniqueness of geometric structure in dimaneion four, and the answer is affirmative. In fact, the homotopy type completely determines the geometry, just as in dimension three:

Theorem 2.1 ([17, §10]) *Assume that M_1 is an X-manifold and M_2 is a Y-manifold. If M_1 is homotopy equivalent to M_2, then $X = Y$.* □

2. Seifert Bundles

The structure of Seifert bundle natually occurs to geometric manifolds. Suppose that we have a geometry X with structure group G, and H is a normal subgroup of G. Let Γ be discrete in G and acts freely on X. Assume that the image Γ' of the projection map $\Gamma \to G/H$ is also discrete. In most cases, Γ is a lattice and $\Gamma \cap H$ is also a lattice (see section 3). There is a fibration sequence: $(\Gamma \cap H)\backslash X \to \Gamma\backslash X \to \Gamma'\backslash(X/H)$.

Normally Γ' does not act freely on X/H. Therefore, the fibration is an orbifold fibration. However, $\Gamma\backslash X$ is itself smooth. Hence we come to the following definition:

Definition 2.1 *A Seifert bundle is an orbifold fibration $M \to B$ in which M is smooth as orbifold.*

In dimension three, the notion of Seifert bundle is used to denote those which the general fiber is S^1. Seifert invariants and Euler number can be defined on a Seifert bundle ([12, §3]). 3-dimensional Seifert bundles on geometric 2-orbifolds are all geometric, and the geometric type is classified by the Euler number and the geometry of the base orbifold ([12, theorem 5.3]).

Analogous to dimension three, some four-manifolds have the structure of Seifert bundles with torus fiber over 2-orbifolds. Seifert invariants and Euler numbers (which consists of two numbers) can be analogously defined. [15] and [16] gave a detailed introduction to 4-dimensional Seifert torus bundles and their geometry types. Note that in dimension four, not all Seifert torus bundle have geometric structure ([16, Theorem B]).

[14] gives an introduction to orbifolds and orbifold fibrations. Seifert bundles behave well with respect to pullback. When the base manifold is smooth, a Seifert bundle is an ordinary fiber bundle.

3. Monodromy and Euler Numbers of Torus Bundles

Given a torus bundle $T^n \to M \to B$, we have an exact sequence of fundamental group: $0 \to \pi_1(T^n) \to \pi_1(M) \to \pi_1(B) \to 1$. Conjugation induces an action of $\pi_1(B)$ on $\pi_1(T^n)$. If for every $g \in \pi_1(B)$, the action of g on $\pi_1(T^n)$ has determinant 1, then we say that M is an **oriented torus bundle**.

Remark: Since our definition of geometry requires that the structure group be connected (only contains orientation-preserving ones), all torus bundles arising in geometric manifolds are orientable. The reader can check case by case when we deal with respective geometries in section 3 and 4. In section 3, we can always take a double cover to make the bundle orientable, so this is not a serious problem. For these considerations, in this article we implicitly assume that all the torus bundles we meet are orientable.

Choose a basis of $\pi_1(T^n)$, and we have a homomorphism $\pi_1(B) \to SL_n(\mathbb{Z})$. This homomorphism is called **Monodromy**.

When $B = S^1$, monodromy has another name: **glueing matrix**. A T^n-bundle over S^1 can be regarded as a quotient of $T^n \times [0, 1]$ under an equivalence relation $(x, 0) \sim (Ax, 1)$. The matrix $A \in SL_n(\mathbb{Z})$ coincides with the monodromy of M.

When B is a surface, we can define the notion of Euler numbers to M. We start with the presentation of $\pi_1(M)$: let $\pi_1(B, p) = \langle a_i, b_i | \prod[a_i, b_i] = 1\rangle$ be the standard presentation of $\pi_1(B)$, $(u_1 \ldots u_n)$ be a basis of $\pi_1(T^2)$, and A_i, B_i be the monodromy matrices along a_i, b_i. Arbitrarily choose lifts $\bar{a}_i, \bar{b}_i \in \pi_1(M)$ that are projected to a_i, b_i. Then $\pi_1(M)$ has presentation

$$\begin{aligned}\pi_1(M) =< u_i, \bar{a}_i, \bar{b}_i | [u_i, u_j] = 1,\ & \bar{a}_i \left(u_1 \ldots u_n\right) \bar{a}_i^{-1} = \left(u_1 \ldots u_n\right) A_i, \\ & \bar{b}_i \left(u_1 \ldots u_n\right) \bar{b}_i^{-1} = \left(u_1 \ldots u_n\right) B_i,\ \Pi[\bar{a}_i, \bar{b}_i] = \Pi u_i^{-e_i} >\end{aligned}$$

for some $e_1 \ldots e_n \in \mathbb{Z}$. The numbers (e_i) are called the Euler numbers of M.

For circle bundles over surfaces (i.e. $n = 1$), the minus Euler number is called the b-invariant, which is explained as the obstruction to the existence of global sections. M can be regarded as the unit circle bundle of a complex line bundle E. The Euler number of M is the same as the Chern class of E.

When $n > 1$, the Euler numbers is not a good "invariant". It depend on many things: the choice of the base point p, the choice of the basis of $\pi_1(F_p)$, and the choice of the lifts of a_i, b_i. Changing the basis of $\pi_1(F_p)$ results in a contravariant change of the Euler numbers. Defining the Euler numbers needs a base point because of monodromy: there is no uniform way to define coordinate systems on every fiber. Changing the lifts of a_i and b_i (such as replacing a_i by $a_i l$) results in a very complicated change of the Euler numbers.

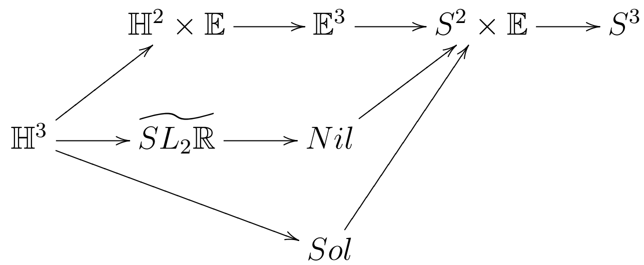
The Euler numbers, as elements in $\mathbb{Z}^n$, is subject to $SL_n(\mathbb{Z})$-actions as we change the presentation of $\pi_1(M)$. However, whether the Euler numbers is 0 or not is a well-defined invariant, independent of the choice of all things.

When all monodromy matrices are trivial, things are much more simple. In this case, a torus bundle is just a $S^1 \times \ldots \times S^1$-bundle, and different factors does not affect each other. Thus we have separated Euler numbers with respect to the n factors.

One important property (when the monodromy is trivial) is the splitting of M. If the Euler numbers have the form $(a, 0 \ldots 0)$, then M splits as $N \times T^{n-1}$. If we wish to rigorously explain this, then we can do as below. For simplicity, we use the case $n = 2$ as an example. Choose local trivializations of M: $M|_{U_i} \cong U_i \times T^2$. Since the monodromy is trivial, by choosing a particular basis on each trivilization, we can make the transition map split: $M|_{U_i} \cong U_i \times S^1 \times S^1$, and $\varphi_{ij} = \alpha_{ij} \times \beta_{ij}$. α_{ij} (resp. β_{ij}) is the transition map of some circle bundle N_1 (resp. N_2), and M is isomorphic to the pullback $N_1 \times_B N_2$. Suppose that the Euler numbers have the form $(a, 0)$, then N_2 is trivial, i.e. $N_2 = B \times S^1$. Therefore $M = N_1 \times S^1$.

4. Three-dimensional Domination Diagram

Recall Diagram 1-2:



Proposition 2.2 *The arrows in Diagram 1-2 are indeed domination relations.*

Proof:

All geometries dominate S^3 since all manifolds dominate S^3.

$Sol \to S^2 \times \mathbb{E}$ can be proved as follows: Let M be a Sol-manifold, then M is a T^2-bundle over S^1. We can find a section s of M and a tubular neighborhood $U \supset s$ such that the intersection of U with each fiber is homeomorphic to a disk. By contracting the complement of this disk on every fiber, we get a map from M to a S^2-bundle over S^1, which must be isomorphic to the trivial bundle $S^2 \times S^1$.

$\widetilde{SL_2\mathbb{R}} \to Nil$ can be proved as follows: A $\widetilde{SL_2\mathbb{R}}$-manifold M is a Seifert bundle over a hyperbolic 2-orbifold B with nonzero Euler number. We first take a finite cover $g : B' \to B$ with B' smooth ([12, Theorem 2.5]). The pull-back bundle $M' = g^*M$ is then a circle bundle over a smooth hyperbolic surface, whose Euler number is an integral multiple of M. By doing vertical pinches ([10, Section 2]), we can produce a degree-one map $M' \to N$ where N is a circle bundle over T^2 with the same Euler number as M' (nonzero). Therefore N is a Nil-manifold. Using the same method, these arrows can be proved: $Nil \to S^2 \times \mathbb{E}$, and $\mathbb{H}^2 \times \mathbb{E} \to \mathbb{E}^3 \to S^2 \times \mathbb{E}$.

Finally, $\mathbb{H}^3$ dominates all other geometries ([6, Theorem 1.1]). $\square$

Proposition 2.3 *The non-arrows in Diagram 1-2 are not domination relations.*

We will leave this proposition to be proved in section 8; there our previous techniques can be used.

3 Typical Covering Manifolds

The **typical covering manifolds** for a geometry X is a set $\mathcal{A}$ of X-manifolds, such that every X-manifold M has a finite cover $M' \to M$ such that $M' \in \mathcal{A}$. For example, a torus is a typical covering manifold for $\mathbb{E}^4$ by Biberbach Theorem ([9, Theorem 8.26]). Note that the set of typical covering manifolds is not unique.

Table 3-1 below lists examples of typical covering manifolds for some 4-dimensional geometries. In the lists and also in the remaining parts of this passage, we always assume that $g > 1$ in the expression "Σ_g".

from now on, when talking about the geometries in Table 3-1, we make the convention that the typical covering manifolds for them are those described in the table.

	Geometry	Typical Covering Manifolds
(1)	$\mathbb{E}^4$	T^4
(2)	$\mathbb{H}^3 \times \mathbb{E}$	($\mathbb{H}^3$-manifolds)$\times S^1$
(3)	$S^3 \times \mathbb{E}$	$S^3 \times S^1$
(4)	$S^2 \times \mathbb{E}^2$	$S^2 \times T^2$
(5)	$\mathbb{H}^2 \times \mathbb{E}^2$	$\Sigma_g \times T^2$ $(g > 1)$
(6)	$\mathbb{H}^2 \times S^2$	$\Sigma_g \times S^2$ $(g > 1)$
(7)	$\widetilde{SL_2\mathbb{R}} \times \mathbb{E}$	($\widetilde{SL_2\mathbb{R}}$-manifolds) $\times S^1$
(8)	$Nil \times \mathbb{E}$	(Nil-manifolds) $\times S^1$
(9)	$Sol \times \mathbb{E}$	(Sol-manifolds) $\times S^1$
(10)	$S^2 \times S^2$	$S^2 \times S^2$

Table 3-1: typical covering manifolds for some geometries

The notion of typical covering manifolds works as follows. In order to prove $X \to Y$, we only need to prove that every typical covering X-manifold dominates some Y-manifold. If we wish to prove $X \nrightarrow Y$, then we only need to prove that every typical covering X-manifold does not dominate any typical covering Y-manifold. This idea leads to Proposition 5.1 and 6.2, which is our basic consideration for proving the diagram.

The following theorems on the discrete subgroups of Lie groups are very useful. A discrete subgroup Γ of a Lie group G is called a **lattice** if $\Gamma \backslash G$ has finite volume.

Lemma 3.1 ([9, Theorem 3.1]) *If G is solvable and $\Gamma \subset G$ is a lattice, then $\Gamma \backslash G$ is compact.* □

Lemma 3.2 ([9, Theorem 3.3]) *Let G be solvable and N be its nilradical. If Γ is a lattice of G, then $\Gamma \cap N$ is a lattice of N, and $\Gamma/(\Gamma \cap N)$ is a lattice of G/N.* □

Lemma 3.3 ([9, Corollary 2.3-1]) *Let G be nilpotent and G' be its commutator subgroup. If Γ is a lattice in G, then $\Gamma \cap G'$ is a lattice in G'.* □

Lemma 3.4 ([9, Theorem 8.27]) *Let G be connected and N be the radical of G, such that G/N has no compact component. If Γ is a lattice in G, then $\Gamma \cap N$ is a lattice in N.* □

Proof of Table 3-1:

In the following paragraphs, we fix our notation: $\Gamma \cong \pi_1(M)$ is a discrete subgroup of the structure group of X, with $M = \Gamma\backslash X$ compact. Denote by G the structure group of X. $\Gamma\backslash G$ is a principle bundle over M, and the fiber is the point stablizer of X. Hence $\Gamma\backslash G$ is compact.

(1) $X = \mathbb{E}^4$:

This follows from Bieberbach Theorem ([9, Theorem 8.26]) which says that every compact flat manifold is covered by flat torus. □

(2) $X = \mathbb{H}^3 \times \mathbb{E}$:

In this case, $\Gamma \subset Isom^0(\mathbb{H}^3) \times Isom^0(\mathbb{E}) =: G_1 \times \mathbb{R}$. From Lemma 3.4 we know that $\Gamma \cap \mathbb{R}$ is a lattice; assume that $s \in \mathbb{R}$ such that $\Gamma \cap \mathbb{R} = \mathbb{Z}s$. Let H be the image of the projection map $\Gamma \hookrightarrow G_1 \times \mathbb{R} \to G_1$; H is also discrete. There is an exact sequence

$$0 \to \mathbb{Z}s \to \Gamma \to H \to 0$$

which expresses M as a Seifert bundle (with circle fiber) over the hyperbolic orbifold $H/\mathbb{H}^3$. Every hyperbolic 3-orbifold has a finite smooth cover. To prove this, we needd only look at the remark below ([12, Theorem 2.5]), and replace the group $PSL_2(\mathbb{R})$ by $PSL_2(\mathbb{C})$.

By taking pull-back, we have a finite cover $M' \to M$ such that M' is a circle bundle over a hyperbolic 3-manifold N. There is an exact sequence: $0 \to \mathbb{Z}s \to \pi_1(M') \to \pi_1(N) \to 0$, where $\pi_1(N) \subset G_1$.

Let Σ be any immersed surface in N and let j be the immersion map. Let $\{a_i, b_i\}$ be a set of standard generator of $\pi_1(\Sigma)$ and let $a'_i = j_*(a_i)$, $b'_i = j_*(b_i)$. Clearly $\prod[a'_i, b'_i] = 1 \in \pi_1(N)$. Let $\bar{a}_i$, $\bar{b}_i$ be any lift of a'_i, b'_i in $\pi_1(M')$; the Euler number $e(j^*M')$ satisfies the relation $\prod[\bar{a}_i, \bar{b}_i] = s^{-e(j^*M')}$. But since $\pi_1(M') \subset G \times \mathbb{R}$, $\bar{a}_i$ and $\bar{b}_i$ can be written as (a'_i, s^{x_i}) and (b'_i, s^{y_i}), and we immediately have $e(j^*M') = 0$.

Let e be the Euler class of M', i.e. the Euler class of the line bundle $E = M' \times_{S^1} \mathbb{C}$. By the above paragraph, e is mapped to zero under the map $H^2(N, \mathbb{Z}) \to Hom(H_2(N, \mathbb{Z}), \mathbb{Z})$. Hence e lies in $Ext(H_1(N, \mathbb{Z}), \mathbb{Z})$ in the Universal Coefficient Theorem. Now we know that e is torsion.

There exists an integer n such that $ne = 0$. This implies that $\pi_1(M')$ is an index n subgroup of another group that is isomorphic to $\pi_1(N) \times \mathbb{Z}(s/n)$. The following lemma shows that there exists a subgroup of $\pi_1(M')$ isomorphic to $\pi' \times \mathbb{Z}$ ($\pi' \subset G_1$) and has index no greater than n. Now we find a finite cover of M' which is a product of $\mathbb{H}^3$-manifold and S^1. □

Lemma 3.5 *Let $H \subset G_1 \times G_2$ be a subgroup such that the projection of H to G_i are both surjective. Let $H_i = H \cap G_i$, then $H/(H_1 \times H_2) \cong G_1/H_1 \cong G_2/H_2 \cong (G_1 \times G_2)/H$.*

Proof: The kernel of the projection map $H \to G_1$ is $H \cap G_2 = H_2$, so we have $G_1 = H/H_2$ and $G_1/H_1 = H/(H_1 \times H_2)$. This establishes the first isomorphism. The second isomorphism is analogous. Four the last isomorphism, we demote $I = G_1/H_1$. The first two isomorphisms show that $H/(H_1 \times H_2) \subset G_1/H_1 \times G_2/H_2 \cong I \times I$ is precisely the graph of some automorphism $f : I \to I$. Thus $(G_1 \times G_2)/H = (G_1/H_1 \times G_2/H_2)/(H/(H_1 \times H_2)) = (I \times I)/\text{graph}(f) \cong I$. □

(3) $X = S^2 \times \mathbb{E}^2$:

The structure group is $SO(3) \times Iso^0(\mathbb{E}^2)$. $\Gamma \cap SO(3) = 1$ because Γ acts without fixed point on X. Thus the projection $\Gamma \to Iso^0(\mathbb{E}^2)$ induces an isomorphism of Γ with its image. We can check that the image must be discrete and has compact quotient. This expresses M as a S^2-bundle over the T^2. All S^2-bundles over T^2 are trivial. $\square$

(4) $X = S^3 \times \mathbb{E}$: Analogous to (3), a $S^3 \times \mathbb{E}$-manifold M is homeomorphic to the mapping torus of an automorphism of spherical 3-manifold. Any self-homeomorphism of a space lifts to its universal cover, hence M is finitely covered by a S^3-bundle over S^1, which is isomorphic to $S^3 \times S^1$. $\square$

(5): $X = \mathbb{H}^2 \times \mathbb{E}^2$:

The structure group here is $G = Iso(\mathbb{H}^2) \times Iso(\mathbb{E}^2)$. By Lemma 3.4, $\Gamma' = \Gamma \cap Iso(\mathbb{E}^2)$ is a lattice, hence must be a rank two translation group. This expresses M as a Seifert torus bundle over a hyperbolic 2-orbifold. Taking a finite cover we can assume that the base orbifold B is smooth; now M is an ordinary torus bundle. The monodromy matrices lie in $SO(2) \cap SL(\Gamma')$ and is a finite group. The kernel of monodromy $\pi' = \ker(\pi_1(B) \to SL(\Gamma'))$ is of finite index. We take the covering space of B with respect to this subgroup, and also take a cover of M by pullback (and replace the old M by the new one). Now lift the standard generators of $\pi_1(B)$ to G, they generates a discrete subgroup $H \subset G$. We have $\Gamma = \Gamma' \times H$, and hence $M = \Sigma_g \times T^2$. $\square$

(6) $X = \mathbb{H}^2 \times S^2$:

Analogous to (3). $\square$

(7) $X = \widetilde{SL_2\mathbb{R}} \times \mathbb{E}$:

Here $G = (\widetilde{SL_2\mathbb{R}} \times_{\mathbb{Z}} \mathbb{R}) \times \mathbb{R}$. The radical of G is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. Lemma 3.4 works in this case, and we have $\Gamma \cap \mathbb{R}^2$ is a lattice. This expresses M as a Seifert torus bundle over hyperbolic 2-orbifolds. Taking finite cover we assume that the base orbifold B is smooth, and M is a ordinary torus bundle. Since $\mathbb{R}^2$ is the center of G, the monodromy is trivial. The Euler numbers must be nonzero (otherwise M would be a $\mathbb{H}^2 \times \mathbb{E}^2$-manifold). By changing the basis of $\pi_1(F)$ (where F denotes the fiber), we can assume that the Euler number of M is of the form $e(M) = (e_1, 0)$. Then M splits as $M = N \times S^1$ where N is the circle bundle over B with Euler number equal to e_1. N is clearly a $\widetilde{SL_2\mathbb{R}}$-manifold. $\square$

(8) $X = Nil \times \mathbb{E}$:

Here $\Gamma \subset G = Iso^0(Nil \times \mathbb{E}) = (Nil \rtimes S^1) \times \mathbb{R}$. The nilradical of $Iso^0(Nil \times \mathbb{E})$ is $N' = Nil \times \mathbb{R}$. Hence $\Gamma \cap N'$ is a lattice (Lemma 3.2). $\Gamma/(\Gamma \cap N')$ is a lattice in S^1 and is finite. Passing to a finite cover, we can assume that $\Gamma \subset N'$.

The commutator subgroup of N' is $Z(Nil) = \mathbb{R}$. $\Gamma \cap \mathbb{R}$ is a lattice and so is $\Gamma' = \Gamma/(\Gamma \cap \mathbb{R}) \subset N'/Z(Nil) = \mathbb{R}^2 \times \mathbb{R}$. We claim that the intersection of Γ with $N = Z(N') = \mathbb{R} \times \mathbb{R}$ is also a lattice. The proof is as follows: Let $\Gamma \cap \mathbb{R} = \mathbb{Z}z_0$ and $e_i = (x_i, y_i, t_i)$ be a basis of $\Gamma/(\Gamma \cap \mathbb{R})$. Let $\bar{e}_i \in \Gamma \cap N'$ be arbitrary elements that projects to e_i. By simple calculation, $[\bar{e}_1, \bar{e}_2] = \begin{pmatrix} 1 & 0 & x_1y_2 - x_2y_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; thus $j_3 = x_1y_2 - x_2y_1$ must be an integral multiples of z_0. Similarly define j_1 and j_2, they are all integral multiple of z_0. Since $\{e_i\}$ is a basis, one of the j_i, say j_3, is nonzero. Cramer Rule tells us that $(j_3/z_0)e_3 - (j_2/z_0)e_2 + (j_1/z_0)e_1$ has the form $(0, 0, *)$. Note that we have already proved that $\Gamma \cap N$ is a lattice.

We have the following exact sequence that gives a torus bundle structure to M:

$$0 \to \mathbb{Z}^2 (= \Gamma \cap N) \to \Gamma \to \mathbb{Z}^2 (= \Gamma/(\Gamma \cap N)) \to 0$$

By definition, N is the center of N', hence the monodromy is trivial. Choose the base (l, h) of $\Gamma \cap N$ such that $l \in Z(Nil)$. The Euler numbers now have the form $(*, 0)$, and this shows that M splits as a product of S^1 and a Nil-manifold. $\square$

The following statement is now trivial (but will be used later):

Proposition 3.1 *Let M be a typical covering manifold of $Nil \times \mathbb{E}$. Then M is a T^3-bundle over S^1. Furthermore, every finite cover of M is again a typical covering manifold.*

Remark: The glueing map for M is of the form:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}$$

(9) $X = Sol \times \mathbb{E}$:

The structure group of $Sol \times \mathbb{E}$ is $Sol \times \mathbb{R}$. $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ (where $\mathbb{R}$ is the direct summand and $\mathbb{R}^2 \subset Sol$) is the nilradical of $Sol \times \mathbb{E}$. By Lemma 3.2, $\Gamma' = \Gamma \cap \mathbb{R}^3$ is a lattice, and the image of projection $\Gamma \hookrightarrow Sol \times \mathbb{R} \to (Sol \times \mathbb{R})/\mathbb{R}^3 \cong \mathbb{R}$ is equal to $\mathbb{Z}t$ for some $t \in \mathbb{R}$. Assume that $\bar{t} \in Sol$ is any element that projects to t.

For any $v = (x, y, z) \in \Gamma'$, $\bar{t}v\bar{t}^{-1} - v = ((x, y)(\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} - I), 0)$. As the choice of v is arbitrary, $\Gamma' \cap \mathbb{R}^2$ ($\mathbb{R}^2 \subset Sol$) must be a lattice in $\mathbb{R}^2$. Therefore we can choose a set of basis $v_1 = (v_{11}, v_{12}, 0), v_2 = (v_{21}, v_{22}, 0), v_3 = (v_{31}, v_{32}, v_{33})$ of Γ'. Under the basis $\{v_1, v_2\}$, conjugation by $\bar{t}$ is represented by an integral Anosov matrix C.

There exists $d_1, d_2 \in \mathbb{Z}$ such that $\bar{t}v_3\bar{t}^{-1} = v_3 + d_1v_1 + d_2v_2$ since the conjugation $\bar{t}(\_)\bar{t}^{-1}$ preserves Γ'. From this formula we get

$$\begin{pmatrix} v_{31} \\ v_{32} \end{pmatrix} = (\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} - I)^{-1}(d_1v_1 + d_2v_2) = \begin{pmatrix} v_1 & v_2 \end{pmatrix} (C - I)^{-1} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

Thus there exists integer numbers a, b, c such that $v_4 = av_1 + bv_2 + cv_3 = (0, 0, u)$ lies in the $\mathbb{R}$ summand of $Sol \times \mathbb{R}$. The subgroup Γ_1 generated by $\{v_1, v_2, v_4, \bar{t}\}$ is an index c subgroup of Γ. Furthermore, the conjugation by $\bar{t}$ on (v_1, v_2, v_4) is represented by the matrix $A = \mathrm{diag}(C', 1)$ where C' is an Anosov matrix. Now let M' be the covering space corresponding to Γ_1, and we know that M' is a product of a Sol-manifold and S^1. $\square$

Sometimes we need to know the structure of all $Sol \times \mathbb{E}$-manifolds. Already we have:

Proposition 3.2 *Any $Sol \times \mathbb{E}$-manifold has the structure of a T^3-bundle over S^1. In the basis of (v_1, v_2, v_3) described above, the glueing matrix is of the form:*

$$\begin{pmatrix} \multicolumn{2}{c}{C} & a \\ & & b \\ 0 & 0 & 1 \end{pmatrix} \ (C \textit{ is Anosov})$$

(10) $X = S^2 \times S^2$:

$Iso^0(S^2 \times S^2) = SO(3) \times SO(3)$ is compact. $\square$

4 Manifolds of $Nil^4, Sol^4_{m,n}, Sol^4_0$ and Sol^4_1 Geometry

In this section, we will determine the structure of Nil^4, $Sol^4_{m,n}$, Sol^4_0 and Sol^4_1-manifolds. One common point of these geometries is that their point stablizers are trivial, i.e. they are the structure group of themselves ([2, Section 6.5]). Manifolds of these geometries are bundles over S^1 with T^3 or Nil-manifold fiber.

1. Nil^4-manifolds

By definition, $Nil^4 = \mathbb{R}^3 \rtimes_A \mathbb{R}$, in which

$$A(t) = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

Nil^4 is nilpotent. Denote by $e_1, e_2, e_3; t$ the standard coordinates; the commutator subgroup is $N = [Nil^4, Nil^4] = \mathbb{R}(e_1, e_2)$. By lemma 3.3, if Γ is a lattice in Nil^4, then $\Gamma \cap N$ is again a lattice, and the quotient $\Gamma' = \Gamma/\Gamma \cap N$ is a lattice in $Nil^4/N = \mathbb{R}^2$.

We can get more information about $\Gamma \cap N$. Suppose $v = (v_1, v_2, 0; 0) \in \Gamma \cap N$, then for any $h \in \Gamma'$, $hvh^{-1} - v$ lies in $\mathbb{R}e_1$. There must exist a h such that $hvh^{-1} \neq v$; hence $\Gamma \cap \mathbb{R}e_1$ is nonzero. We can therefore assume that $\Gamma \cap N$ is generated by $l = (l_1, 0, 0; 0)$ and $h = (h_1, h_2, 0, 0)$.

Assume $x = (x_1, x_2, x_3; x_4), y = (y_1, y_2, y_3; y_4)$ are two elements of Γ that project to a basis of Γ' (i.e. $x_3y_4 - x_4y_3 \neq 0$). By concrete calculation we have the following relation:

$$x \begin{pmatrix} l & h \end{pmatrix} x^{-1} = \begin{pmatrix} l & h \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \text{ (for some } \lambda \in \mathbb{Z})$$

The same relation holds for y. Changing the basis elements x, y we can assume that x commutes with l, h. This implies $x_4 = 0$. Finally we calculate $[y, x]$:

$$[y, x] = (x_2y_4 + x_3y_4^2/2,\ x_3y_4,\ 0; 0)$$

we find that $[x, y] = l^a h^b$ for $a, b \in \mathbb{Z}, b \neq 0$.

The exact sequence $0 \to \mathbb{Z}(l, h, x) \to \Gamma \to \mathbb{Z}y \to 0$ gives M the structure of T^3-bundle over S^1. To summarize, we have:

Proposition 4.1 *If M is a Nil^4-manifold, then M is a T^3-bundle over S^1 with glueing matrix*

$$A = \begin{pmatrix} 1 & \lambda & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

such that $\lambda \neq 0$, $b \neq 0$. □

Calculating the homology groups of M, we have:

Proposition 4.2 *$H_1(M, \mathbb{Z}) = \mathbb{Z}^2 \oplus (torsion)$, $H_2(M, \mathbb{Z}) = \mathbb{Z}^2 \oplus (torsion)$, $H_3(M, \mathbb{Z}) = \mathbb{Z}^2$.*

Proof: We use the following mapping torus long exact sequence; its proof can be found in [3, Section 2.2, Example 2.48].

$$H_3(T^3,\mathbb{Z}) \xrightarrow{0} H_3(T^3,\mathbb{Z}) \longrightarrow H_3(M,\mathbb{Z}) \longrightarrow H_2(T^3,\mathbb{Z}) \xrightarrow{1-\Lambda^2 A} H_2(T^3,\mathbb{Z}) \to$$

$$\to H_2(M,\mathbb{Z}) \longrightarrow H_1(T^3,\mathbb{Z}) \xrightarrow{1-A} H_1(T^3,\mathbb{Z}) \longrightarrow H_1(M,\mathbb{Z}) \to \mathbb{Z} \to 0$$

Let x, y, z be a basis of $H_1(T^3,\mathbb{Z})$, such that A is represented by the matrix in Proposition 4.1.

We have $coker(1-A) = \mathbb{Z} \oplus (torsion)$ where the $\mathbb{Z}$-summand is generated by z.

Under the basis $\{x \wedge y, x \wedge z, y \wedge z\}$ of $H_2(T^3,\mathbb{Z})$,

$$\Lambda^2 A = \begin{pmatrix} 1 & \lambda & \lambda b - a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Hence $coker(1-\Lambda^2 A) = \mathbb{Z} \oplus (torsion)$ where the $\mathbb{Z}$-summand is generated by $y \wedge z$. Furthermore we have $ker(1-A) = \mathbb{Z}$ which is generated by x. Thus $H_2(M,\mathbb{Z}) = \mathbb{Z}^2 \oplus (torsion)$. $ker(1-\Lambda^2 A) = \mathbb{Z}$ is generated by $x \wedge y$, and hence $H_3(M,\mathbb{Z}) = \mathbb{Z}^2$. □

Remark: From the proof of Proposition 4.1, we can find the generator of the torsion-free parts of the homology groups. $H_1(M,\mathbb{R})$ is generated by two elements: α, being represented by a section of the bundle, and β, a 1-dimensional "subspace" of the T^3 fiber spanned by z, which is actually a S^1. $H_2(M,\mathbb{R})$ is generated by two elementa: γ, a hyperplane in T^3 spanned by $\{y, z\}$, and δ, a subbundle of M — its fiber is a 1-dimensional subspace in T^3 spanned by x. $H_3(M,\mathbb{R})$ is generated by two elements: ϵ, the fiber T^3, and ϕ, a subbundle of M whose fiber is the hyperplane in T^3 generated by $\{x, y\}$.

Using these generators, we can determine the product structure of $H^*(M)$. We will do the calculation in Proposition 5.5 and 6.5(2).

2. $Sol^4_{m,n}$-manifolds

By definition, $Sol^4_{m,n} = \mathbb{R}^3 \rtimes_A \mathbb{R}$, in which

$$A(t) = \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{ct} \end{pmatrix}$$

The three real numbers a, b, c are the three roots of the equation $x^3 - mx^2 + nx - 1 = 0 (m, n \in \mathbb{Z})$ and are required to be nonzero and be different from each other. If two roots have the same value, then there would S^1 action, and the geometry becomes Sol^4_0.

$Sol^4_{m,n}$ is solvable and its nilradical is $\mathbb{R}^3$. If Γ is a lattice, then $\Gamma \cap \mathbb{R}^3$ is a lattice and $\Gamma/(\Gamma \cap \mathbb{R}^3) = \mathbb{Z}t \subset \mathbb{R}$. Hence every $Sol^4_{m,n}$-manifold M can be regarded as a T^3-bundle over S^1. Let A be its glueing map. A is also the operator by which $t(\_)t^{-1}$ acts on the lattice $\Gamma \cap \mathbb{R}^3$. By the definition of $Sol^4_{m,n}$, ± 1 are not eigenvalues of A.

Definition 4.1 *If a (integral/real/complex) matrix A with* $\det(A) = 1$ *does not have eigenvalue λ such that $|\lambda| = 1$, then A is called* ***Anosov****. Such matrix always has* $\det(1-A) \neq 0$.

Proposition 4.3 *$Sol^4_{m,n}$-manifold has the structure of T^3-bundle over S^1 with Anosov glueing matrix $A \in SL_3(\mathbb{Z})$, and A has three different real eigenvalues.* □

Proposition 4.4 *Let M be a $Sol^4_{m,n}$-manifold. Then $H_1(M,\mathbb{R}) = \mathbb{R}$ and $H_2(M,\mathbb{R}) = 0$.*

Proof: Again we use the mapping torus sequence. The sequence has the same form as in Proposition 4.2, but the matrix A here is Anosov. Since $\Lambda^2 A = A^{-1}$, $\Lambda^2 A$ is also Anosov. Hence in real coefficient, $1-A$ and $1-\Lambda^2 A$ are both isomorphisms. The homology groups are now clear. □

3. Sol^4_0-manifolds

By definition, the structure group of Sol^4_0 is $G = \mathbb{R}^3 \rtimes_A (\mathbb{R} \times SO(2))$ where

$$A(t,\theta) = \begin{pmatrix} e^t \cos\theta & e^t \sin\theta & 0 \\ -e^t \sin\theta & e^t \cos\theta & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$$

and the geometry is $Sol^4_0 = G/SO(2)$. Let $\Gamma \subset G$ be discrete and $M = \Gamma\backslash X$ is compact. $\Gamma\backslash G$ is a S^1-bundle over X and hence is compact; therefore Γ is a lattice in G. G is solvable and its nilradical is $N = \mathbb{R}^3$. By Lemma 3.2, $\Gamma' = \Gamma \cap N \subset N$ and $H = \Gamma/\Gamma' \subset G/N = \mathbb{R} \times SO(2)$ are both lattices.

There are only two kinds of lattices in $\mathbb{R} \times SO(2)$: $\mathbb{Z}$ and $\mathbb{Z} \oplus \mathbb{Z}_n$. If $H \cong \mathbb{Z}$, then we suppose that H is generated by (t_0, θ_0), $t_0 \neq 0$. In this case, M is a T^3-bundle over S^1 with glueing matrix $A \in SL_3(\mathbb{Z})$ which is conjugate to

$$\begin{pmatrix} e^{t_0} \cos\theta_0 & e^{t_0} \sin\theta_0 & 0 \\ -e^{t_0} \sin\theta_0 & e^{t_0} \cos\theta_0 & 0 \\ 0 & 0 & e^{-2t_0} \end{pmatrix}$$

If $H \cong \mathbb{Z} \oplus \mathbb{Z}_n$, then suppose that H is generated by $a = (t,\theta)$, $t \neq 0$, and $b = (0,\phi)$, where $\phi = 2\pi/n$. The conjugation action by a and b both preserves $\Gamma' \cong \mathbb{Z}^3$ and are isomorphisms on it. They are represented by matrices

$$A = \begin{pmatrix} e^t \cos\theta & e^t \sin\theta & 0 \\ -e^t \sin\theta & e^t \cos\theta & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}, B = \begin{pmatrix} \cos(2\pi/n) & \sin(2\pi/n) & 0 \\ \sin(2\pi/n) & \cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let (x,y,z) be standard coordinates in $\mathbb{R}^3$. We claim that Γ' has non-empty intersection with the z-axis. Otherwise, there must exist $v = (v_1, v_2, v_3) \in \Gamma'$ such that $(v_1, v_2) \neq (0,0)$ and $v_3 \neq 0$. However, $v + Bv + B^2 v + \ldots + B^{n-1} v$ is nonzero and must lie in the z-axis, which is a contradiction.

Clearly z is an eigenvector of A with eigenvalue e^{-2t}. However, the z-axis has non-empty intersection with Γ', which implies that e^{-2t} must be an integer, and since $t \neq 0$ it is not equal to 1. $\det(A)$, which is an integer, is divisible by e^{-2t}, hence cannot be 1. This leads to a contradiction, and hence $H \cong \mathbb{Z} \oplus \mathbb{Z}_n$ cannot happen.

Summarizing our results, we have:

Proposition 4.5 *Sol^4_0-manifold has the structure of T^3-bundle over S^1 with Anosov glueing matrix $A \in SL_3(\mathbb{Z})$. A either has one real and two complex eigenvalues, or has three real eigenvalues in which two are equal.* □

For the converse statements, we have:

Proposition 4.6 *Let M be a T^3-bundle over S^1 with glueing matrix A, and A is an Anosov matrix. Then M has $Sol^4_{m,n}$ or Sol^4_0 structure.*

Proof: A either has ①: three distinct real eigenvalues, or ②: a double eigenvalue, or ③: one real and two complex eigenvalues.

①: If A has three real eigenvalues $a \neq b \neq c$, then suppose its characteristic polynomial is $p_A(x) = x^3 - mx^2 + nx - 1$, where we let the numbers m, n be the same as the numbers m, n in $Sol^4_{m,n}$. A is diagonalizable: $P^{-1}AP = diag(a, b, c)$ for some real matrix P. Let $v_i (i = 1, 2, 3)$ be the column vectors of P, and the lattice $\Gamma = (jv_1 + kv_2 + lv_3 + m) | j, k, l, m \in \mathbb{Z} \in Sol^4_{m,n}$ has $\Gamma \backslash Sol^4_{m,n} = M$.

③ can be proved by the same method as ①. ② is just a special case of ③. □

Proposition 4.7 *If M is a Sol^4_0-manifold, then $H_1(M, \mathbb{R}) = \mathbb{R}$ and $H_2(M, \mathbb{R}) = 0$.*

Proof: it is the same as Proposition 4.4 □

4. Sol^4_1-manifolds

By definition, $Sol^4_1 = Nil \rtimes_A \mathbb{R}$, where

$$A(t) : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & e^t x & z \\ 0 & 1 & e^{-t} y \\ 0 & 0 & 1 \end{pmatrix}$$

Note that the center of Sol^4_1 is spanned by $z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Sol^4_1 is solvable and its nilradical is Nil. If $\Gamma \subset Sol^4_1$ is a lattice, then by Lemma 3.2, $\Gamma' = \Gamma \cap Nil$ is also a lattice, and $\Gamma/\Gamma' = \mathbb{Z}l \subset Sol^4_1/Nil = \mathbb{R}$. Let $L \in \Gamma$ be any element which projects to l.

The structure of Γ' is quite clear. The group Nil has an exact sequence $0 \to \mathbb{R} \to Nil \to \mathbb{R}^2 \to 0$, where the first term $\mathbb{R}$ is spanned by z. There is also an exact sequence of Γ' which is compatible with that of Nil:

$$0 \to \Gamma' \cap \mathbb{R} (\cong \mathbb{Z}) \to \Gamma' \to \Gamma'/(\Gamma' \cap \mathbb{R})(\cong \mathbb{Z}^2) \to 0$$

This expresses $E = \Gamma' \backslash Nil$ as a circle bundle over T^2 (with nonzero Euler number).

The conjugation action by L induces an isomorphism of Γ. Since $[z, L] = 1$, this isomorphism respects the exact sequence of Γ':

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma' \cap \mathbb{R} & \longrightarrow & \Gamma' & \longrightarrow & \Gamma'/(\Gamma' \cap \mathbb{R})(\cong \mathbb{Z}^2) & \longrightarrow & 0 \\ & & \downarrow = & & \downarrow & & \downarrow C & & \\ 0 & \longrightarrow & \Gamma' \cap \mathbb{R} & \longrightarrow & \Gamma' & \longrightarrow & \Gamma'/(\Gamma' \cap \mathbb{R})(\cong \mathbb{Z}^2) & \longrightarrow & 0 \end{array}$$

By looking at the definition of Sol^4_1, we know that C is an Anosov matrix.

$M = \Gamma\backslash Sol_1^4$ is a E-bundle over S^1 with glueing map f. By the above discussion, f can be described as:

$$\begin{array}{ccccc} S^1 & \longrightarrow & E & \longrightarrow & T^2 \\ \downarrow{=} & & \downarrow{f} & & \downarrow{C} \\ S^1 & \longrightarrow & E & \longrightarrow & T^2 \end{array}$$

i.e. F is a bundle isomorphism of E covering an Anosov map of T^2. This describes the structure of Sol_1^4-manifolds.

Summarizing our results:

Proposition 4.8 *Let M be a Sol_1^4-manifold. Then there exists a 3-manifold E which is a circle bundle over T^2 (with nonzero Euler number), and a bundle isomorphism f of E, such that M is a E-bundle over S^1 with glueing map f.* □

The converse conclusion is also correct:

Proposition 4.9 *Let E be a circle bundle over T^2 with Euler number $k \neq 0$. M is a E-bundle over S^1 with glueing map $f \in Aut(E)$, such that f is a bundle isomorphism of E which covers an Anosov map C of T^2. Then M has Sol_1^4 geometry.*

Remark: The isotopy class of bundle isomorphism is classified by $[T^2, S^1] = H^1(T^2, \mathbb{Z})$, and is determined by the induced isomorphism f_* on π_1. What we essentially prove here is that all possible bundle isomorphism f can be realized.

Proof: $\pi_1(M)$ is generated by $\pi_1(E)$ and an additional element K, which satisfies the relation $KgK^{-1} = f_*(g)$ for $g \in \pi_1(E)$.

Let $x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ be standard generators of Nil.

For a real number u, denote by x^u the matrix $\begin{pmatrix} 1 & u & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (y^u, z^u is analogous).

We first embed $\pi_1(E)$ into Nil. Suppose that $\pi_1(E) = \Gamma' \subset Nil$ is generated by $e_1 = x^u y^v$, $e_2 = x^w y^t$ and $e_3 = z$. The real numbers u, v, w, t are to be determined. The Euler number of E is $k (k \neq 0)$ implies that $[e_1, e_2] = z^k$, which is equivalent to $ut - vw = k$.

Let $C = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$. Then we can suppose that $f_*(e_1) = e_1^p e_2^r z^{b_1}$, $f_*(e_2) = e_1^q e_2^s z^{b_2}$ and $f_*(z) = z$. Since C is Anosov,

$$\begin{pmatrix} u' & w' \\ v' & t' \end{pmatrix}^{-1} C \begin{pmatrix} u' & w' \\ v' & t' \end{pmatrix} = \begin{pmatrix} e^l & 0 \\ 0 & e^{-l} \end{pmatrix}$$

for some real numbers u', v', w', t', l. By multiplying with a constant number, we can make $u't' - v'w' = k$ hold. Then we set $u = u', v = v', w = w', t = t'$.

Set $D = \begin{pmatrix} u & w \\ v & t \end{pmatrix}$ and define $\begin{pmatrix} i \\ j \end{pmatrix} = D^{-1} C^{-1} \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. Let $L = (0; l) \in Nil \rtimes \mathbb{R} = Sol_0^4$. Define $K = x^j y^{-i} L$. By direct calculation, we can check that $K(\_)K^{-1}$ is the same as f_*, both being isomorphisms of Γ'. Therefore, the subgroup of Sol_1^4 generated by $\{e_1, e_2, e_3, K\}$ is isomorphic to $\pi_1(M)$. □

Finally we calculate the homology of Sol_1^4-manifolds.

Proposition 4.10 *Let M be a Sol_1^4-manifold. Then $H_1(M,\mathbb{R})=\mathbb{R}$, $H_2(M,\mathbb{R})=0$.*

Proof: The notations E, f has the same meaning as above.

The homology of E can be calculated by the Wang sequence:

$$0 \longrightarrow H^1(T^2,\mathbb{R}) \longrightarrow H^1(E,\mathbb{R}) \longrightarrow H^1(S^1,\mathbb{R}) \xrightarrow[=]{e} H^2(T^2,\mathbb{R})$$

which implies that $H_1(E,\mathbb{R}) = H_2(E,\mathbb{R}) = \mathbb{R}^2$.

Then we apply the mapping torus sequence:

$$H_3(E,\mathbb{R}) \xrightarrow{0} H_3(E,\mathbb{R}) \longrightarrow H_3(M,\mathbb{R}) \longrightarrow H_2(E,\mathbb{R}) \xrightarrow{1-f_*|_{H_2}} H_2(E,\mathbb{R}) \to$$

$$\to H_2(M,\mathbb{R}) \longrightarrow H_1(E,\mathbb{R}) \xrightarrow{1-f_*|_{H_1}} H_1(E,\mathbb{R}) \longrightarrow H_1(M,\mathbb{R}) \to \mathbb{R} \to 0$$

Since f is an bundle morphism, f^* respects the Wang sequence. Hence $f^*|_{H^1} = {}^tC$ and is Anosov. By Poincare duality, $f^*|_{H^2} = C^{-1}$ is Anosov; so does $f_*|_{H_1}$ and $f_*|_{H_2}$. Now $1-f_*|_{H_1}$ and $1-f_*|_{H_2}$ are both isomorphism, and the result follows. $\square$

We have another observation for Sol_1^4-manifolds:

Proposition 4.11 *If M is a Sol_1^4-manifold, then M has the structure of a circle bundle over a Sol-manifold B, and the Euler class is non-torsion.*

Proof:: The bundle structure comes from the circle bundle structure of E. f is a bundle isomorphism of E which covers an Anosov map C of T^2. The projection of E to T^2 is f-equivariant, so induces a global circle bundle structure on M. The base manifold B is a T^2-bundle over S^1 with glueing map C, and hence is a Sol-manifold.

If $e(M)$ is torsion, then M will finitely cover $B \times S^1$, and will have $Sol \times \mathbb{E}$ geometry. This contradicts Theorem 2.1. $\square$

For Nil^4-manifolds, we have anlogous results:

Proposition 4.12 *If M is a Nil^4-manifold, then M has the structure of a circle bundle over Nil-manifold, and the Euler class is non-torsion.*

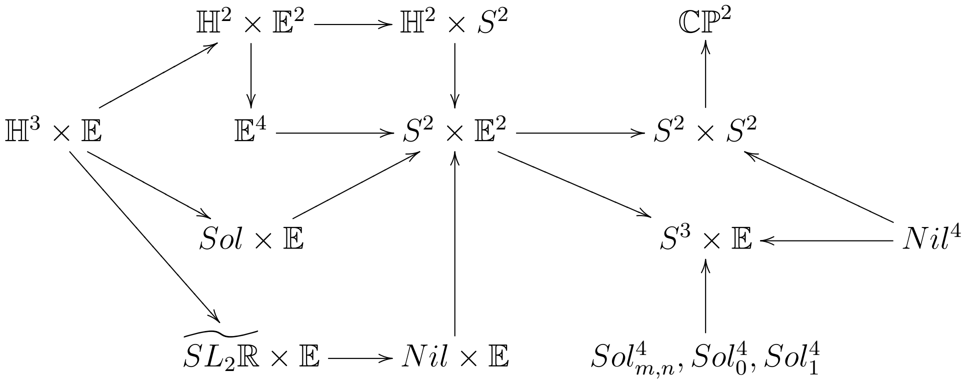
Proof: By Proposition 4.1, M is a T^3-bundle over S^1 with glueing matrix

$$A = \begin{pmatrix} 1 & \lambda & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

such that $\lambda \neq 0$, $b \neq 0$. By projecting to the last two coordinates, M is expressed as a circle bundle over a Nil-manifold. If the Euler class is torsion, then M will have $Nil \times \mathbb{E}$ geometry, which contradicts Theorem 2.1. $\square$

5 Proof of Diagram 1-1 (without $\mathbb{H}^2 \times \mathbb{H}^2$): Arrows

Let us recall Diagram 1-1:



Using the results developed in section 3 and 4, it is now easy to prove the arrows. Our purpose to develop the notion of typical covering manifold is the following statement:

Proposition 5.1 *Let X and Y be geometries. If every typical covering manifold of X dominates some Y-manifold, then $X \to Y$ holds.* $\square$

Then we can begin the proof of arrows:

Proposition 5.2 *(1) $\mathbb{H}^2 \times \mathbb{E}^2 \to \mathbb{H}^2 \times S^2 \to S^2 \times \mathbb{E}^2 \to S^2 \times S^2 \to \mathbb{CP}^2$. (2) $\mathbb{H}^2 \times \mathbb{E}^2 \to \mathbb{E}^4 \to S^2 \times \mathbb{E}^2 \to S^3 \times \mathbb{E}$.*

Proof: Use Proposition 5.1 and Table 3-1.

(1) Since high-genus surface dominates low-genus surface, we clearly have $\Sigma_{g_1} \times T^2$ dominates $\Sigma_{g_1} \times S^2$ dominates $T^2 \times S^2$ dominates $S^2 \times S^2$ dominates $\mathbb{CP}^2$.

(2) It is also clear that $\Sigma_g \times T^2$ dominates $T^2 \times T^2$ dominates $S^2 \times T^2 = S^2 \times S^1 \times S^1$ dominates $S^3 \times S^1$. $\square$

Proposition 5.3 *(1) $\mathbb{H}^3 \times \mathbb{E}$ dominates $\mathbb{H}^2 \times \mathbb{E}^2, Sol \times \mathbb{E}$ and $\widetilde{SL_2\mathbb{R}} \times \mathbb{E}$. (2) $Sol \times \mathbb{E} \to S^2 \times \mathbb{E}^2$. (3) $\widetilde{SL_2\mathbb{R}} \times \mathbb{E} \to Nil \times \mathbb{E} \to S^2 \times \mathbb{E}^2$.*

Proof: This is a direct consequence of Corollary 7.5. $\square$

Proposition 5.4 *$Nil^4, Sol^4_{m,n}, Sol^4_0, Sol^4_1$ dominates $S^3 \times \mathbb{E}$.*

Proof: By Proposition 4.1, 4.3, 4.5 and 4.8, manifolds with $Nil^4, Sol^4_{m,n}, Sol^4_0$ or Sol^4_1 geometry have the structure of a fiber bundle over S^1. Let M be such a manifold. We can find a section s of M and a tubular neighborhood $U \supset s$ such that the intersection of U with each fiber is homeomorphic to a disk. By contracting the complement of this disk on every fiber, we get a map from M to a S^3-bundle over S^1, which must be isomorphic to the trivial bundle $S^3 \times S^1$. $\square$

To determine which geometry dominates $S^2 \times S^2$, we need to develop a technical lemma:

Lemma 5.1 *There exists $f : M \to S^2 \times S^2$ with $\deg(f) \neq 0$ iff $b_2(M) > 1$ and there is a nonzero element $x \in H^2(M, \mathbb{R})$ such that $x^2 = 0$.*

Proof: Let a, b be the standard generators of $H^2(S^2 \times S^2, \mathbb{R})$.

If such f exists, then $f^*(a)f^*(b) = f^*(ab) = \deg(f)[M] \neq 0$ and $f^*(a)^2 = f^*(b)^2 = 0$. This implies that $f^*(a)$ and $f^*(b)$ are linearly independant, and thus $b_2(M) > 1$. a is our desired element x.

Conversely: $x^2 = 0$ means that the intersection form of M is not definite. Hence there exists another nonzero element $y \in H^2(M, \mathbb{R})$ such that $y^2 = 0$ and $xy \neq 0$. x induces a map $f_x : M \to \mathbb{CP}^\infty$ which by cellular approximation, can be homotopied to a map $f'_x : M \to \mathbb{CP}^2$. Since $x^2 = 0$, we have $\deg(f'_x) = 0$. This implies that f'_x can be deformed to a map $f''_x : M \to S^2$. Similarly, we can define f''_y. Consider $f = f''_x \times f''_y : M \to S^2 \times S^2$. We have $\deg(f)[M] = xy$ hence $\deg(f) \neq 0$. □

Proposition 5.5 *$Nil^4 \to S^2 \times S^2$.*

Proof: By Proposition 4.2, a Nil^4 manifold has $b_2 = 2$. □

Now the arrows in Diagram 1-1 are all proven.

6 Proof of Diagram 1-1: Non-arrows (Part 1)

In this section, we prove a part of the non-arrows. Those we wish to prove cannot be derived from lower-dimensional results, and must be proved independently. The proof depends on several technical lemmas:

Lemma 6.1 *If $f : M \to N$ is of nonzero degree, then the image of $f_* : \pi_1(M) \to \pi_1(N)$ is of finite index in $\pi_1(N)$.*

Proof: Let $d = \deg(f)$ and $H = \mathrm{Im}(f_*)$. If $[\pi_1(N) : H] > d$, then there exists a subgroup $H' \subset \pi_1(N)$ such that $\infty > [\pi_1(N) : H'] > d$. Let N' be the (finite) covering space of N such that $\pi_1(N') = H'$, and denote by p the projection map. By knowledges of covering spaces, there exists a map $f' : M \to N'$ such that $f = pf'$. Calculating the degree we get a contradiction. □

Corollary 6.1 *If $g_1 < g_2$, then Σ_{g_1} does not dominate Σ_{g_2}. If S is simply-connected, then $S \times \Sigma_{g_1}$ does not dominate $S \times \Sigma_{g_2}$.*

Proof: If $f : \Sigma_{g_1} \to \Sigma_{g_2}$ has nonzero degree, then by Lemma 6.1, there exists a finite cover $\Sigma_{g_3} \to \Sigma_{g_2}$ and a lifted map $\tilde{f} : \Sigma_{g_1} \to \Sigma_{g_3}$, such that $\tilde{f}_* : \pi_1(\Sigma_{g_1}) \to \pi_1(\Sigma_{g_3})$ is surjective. Obviously $g_3 > g_2$; and $\tilde{f}_* : H_1(\Sigma_{g_1}, \mathbb{Z}) \to H_1(\Sigma_{g_3}, \mathbb{Z})$ is also surjective. This is impossible.

The second conclusion holds for the same reason. □

A $K(G, 1)$ is a CW complex which has contractible universal cover and has fundamental group G; the homotopy type of $K(G, 1)$ is unique. A detailed introduction of $K(G, 1)$ can be found in section 1.B of [3]. If X is a geometry which is not $S^4, S^3 \times \mathbb{E}^2, S^2 \times S^2, S^2 \times \mathbb{E}^2$ or $S^2 \times \mathbb{H}^2$, then every X-manifold is a $K(G, 1)$.

Lemma 6.2 ([3, Proposition 1B.9]) *Let M be a connected CW complex and let N be a $K(G,1)$. Then every homomorphism $\pi_1(M) \to \pi_1(N)$ is induced by a map $M \to N$ which is unique up to homotopy.* $\square$

The fundamental criterion for non-arrows is the following statement. This criterion is much stronger than the definition of non-domination, but still grabs the essence.

Proposition 6.2 *Assume that X, Y are two geometries, and $\mathcal{A}$ (resp. $\mathcal{B}$) is a set of typical covering manifold for X (resp. Y). If for every $M_1 \in \mathcal{A}$ and $M_2 \in \mathcal{B}$, M_1 does not dominate M_2, then $X \nrightarrow Y$.*

Proof: Pick any X-manifold M_1, and assume that M_1 dominates some Y-manifold M_2. M_2 is finitely covered by some $M_2' \in \mathcal{B}$. The pull back $M_1' = M_1 \times_{M_2} M_2'$ is a finite cover of M_1, and is covered by some $M_1'' \in \mathcal{A}$. The composition map $M_1'' \to M_1' \to M_2'$ is of nonzero degree, which contradicts the assumption in our proposition. $\square$

Now we start proving non-arrows. We do not need to prove every non-arrow, since the domination relation is transitive. For example, if we wish to prove that Nil^4 does not dominate any other geometries except those shown in the diagram, we only need to show that it does not dominate $S^2 \times \mathbb{E}^2, Sol_0^4, Sol_{m,n}^4$ and Sol_1^4. For another example, if we have shown that $\mathbb{H}^2 \times \mathbb{E}^2, \mathbb{H}^3 \times \mathbb{E}, Sol_0^4, Sol_{m,n}^4$ and Sol_1^4 does not dominate Nil^4, then there would be no extra arrows pointing toward Nil.

With a little effort, we can check that the propositions proved below, combined with Corollary 8.1, are sufficient to imply all non-arrows.

Proposition 6.3 *(1) $\mathbb{CP}^2 \nrightarrow S^2 \times S^2$, $\mathbb{CP}^2 \nrightarrow S^3 \times \mathbb{E}$. (2) $S^2 \times S^2 \nrightarrow S^3 \times \mathbb{E}$, $S^3 \times \mathbb{E} \nrightarrow \mathbb{CP}^2$.*

Proof:

(1) Note that $\mathbb{CP}^2$ is the only $\mathbb{CP}^2$-manifold — a consequence of the Lefschetz fixed-point formula. The first conclusion is from Lemma 5.1 and $b_2(\mathbb{CP}^2) = 1$. The second conclusion is from $H^1(\mathbb{CP}^2) = 0$ so that $\mathbb{CP}^2$ does not dominate $S^3 \times S^1$. By Table 3-1(3), $S^3 \times S^1$ is the typical covering manifold of $S^3 \times \mathbb{E}$.

(2) Clearly $S^2 \times S^2$ does not dominate $S^3 \times S^1$ and $S^3 \times S^1$ does not dominate $\mathbb{CP}^2$. $\square$

Corollary 6.4 *$S^2 \times S^2$ and $S^3 \times \mathbb{E}$ dominates no other geometry except those indicated in the domination diagram.* $\square$

Proposition 6.5 *(1) $Sol_{m,n}^4, Sol_0^4, Sol_1^4$ does not dominate $\mathbb{CP}^2$. (2) Nil^4 does not dominate $S^2 \times \mathbb{E}^2$. (3) $Nil^4, Sol_{m,n}^4, Sol_0^4, Sol_1^4$ does not dominate each other.*

Proof:

(1) By Proposition 4.4, 4.7, 4.10, $Sol_{m,n}^4, Sol_0^4, Sol_1^4$-manifolds have $b_2 = 0$.

(2) Let M be any Nil^4-manifold. Again we look back at the remark below Proposition 4.2, and use the notations defined there. $H_3(M, \mathbb{R})$ is generated by ϵ, which is represented by the T^3-fiber, and ϕ, which is a subbundle. The intersection of these two chains is a hyperplane in the T^3 fiber, which is spanned by $\{x, y\}$. Denote this chain by c. From the proof of proposition 4.2, we can know that c lies in the image of $1 - A$ (here the coefficient is real numbers), and thus projects to zero in $H_2(M, \mathbb{R})$. Now denote by $\hat{\epsilon}, \hat{\phi} \in H^1(M, \mathbb{R})$

the Poincare dual of ϵ, ϕ, then we have $\hat{\epsilon}\hat{\phi} = 0 \in H^2(M, \mathbb{R})$. Since they generates H^1, for any two elements $u, v \in H^1(M, \mathbb{R})$ we have $uv = 0$.

Suppose that $f : M \to S^2 \times T^2$ is of nonzero degree. Assume that $a, b \in H^1(S^2 \times T^2, \mathbb{R})$ and $c \in H^2(S^2 \times T^2, \mathbb{R})$ are standard generators, then we have $abc = \pm[S^2 \times T^2]$ and $f^*(abc) = (f^*(a)f^*(b))f^*(c) = 0$, thus $\deg(f) = 0$. Finally use Proposition 5.6 and Table 3-1(4).

(3) By Proposition 4.1, 4.3, 4.7 and 4.10, if M is a manifold of $Nil^4, Sol^4_{m,n}, Sol^4_0$ or Sol^4_1 type, then $\pi_1(M)$ has the form $\pi_1(M) = N \rtimes_A \mathbb{Z}$, where N is $\mathbb{Z}^3$ or the fundamental group of a Nil-manifold, and $A \in \mathrm{Aut}(N)$.

Let M_1 be a Sol^4_1-manifold, M_2 be of $Nil^4, Sol^4_{m,n}$ or Sol^4_0 type, and suppose $f : M_1 \to M_2$ be of nonzero degree. By Lemma 6.1, after taking a finite cover of M_2 we can assume that $f_* : \pi_1(M_1) \to \pi_1(M_2)$ is surjective. Assume that $\pi_1(M_1) = N \rtimes_A \mathbb{Z}$ and $\pi_1(M_2) = \mathbb{Z}^3 \rtimes_B \mathbb{Z}$, where N is nilpotent but not abelian. f_* maps N to a normal subgroup of $\pi_1(M_2)$, which must be contained in $\mathbb{Z}^3$. Therefore, f_* factors through $N/[N, N] = \mathbb{Z}^2$. Clearly N cannot be sent surjectively onto $\mathbb{Z}^3$, and hence f_* is not surjective. This implies M_1 cannot dominate M_2. For analogous reason, M_2 does not dominate M_1.

Now let M_1, M_2 both be of $Nil^4, Sol^4_{m,n}$ or Sol^4_0 type, and their geometries are different. Suppose that $f : M_1 \to M_2$ be of nonzero degree. By taking a finite cover of M_2, we can assume that f_* is surjective. Assume that $\pi_1(M_1) = \mathbb{Z}^3 \rtimes_A \mathbb{Z}$ and $\pi_1(M_2) = \mathbb{Z}^3 \rtimes_B \mathbb{Z}$, where A and B are elements in $SL_3(\mathbb{Z})$. f_* takes $\mathbb{Z}^3 \subset \pi_1(M_1)$ into $\mathbb{Z}^3 \subset \pi_1(M_2)$, and must be an isomorphism. Denote this isomorphism by P. Let t be the generator of $\mathbb{Z} \subset \pi_1(M_1)$ and t' the generator of $\mathbb{Z} \subset \pi_1(M_2)$. Then f_* maps t to ut' for some $u \in \mathbb{Z}^3$. For any $v \in \mathbb{Z}^3 \subset \pi_1(M_1)$, we have

$$PAv = f_*(Av) = f_*(tvt^{-1}) = ut'f_*(v)t'^{-1}(-u) = Bf_*(v) = BPv$$

which implies $B = PAP^{-1}$. But M_1 and M_2 has different geometry. Proposision 4.1, 4.3 and 4.5 tell us that A and B cannot be conjugate. Therefore the assumed f does not exist. □

Corollary 6.6 *$Nil^4, Sol^4_{m,n}, Sol^4_0, Sol^4_1$ dominates no other geometry except those indicated in the domination diagram.* □

Proposition 6.7 *(1) $S^2 \times \mathbb{E}^2 \nrightarrow \mathbb{H}^2 \times S^2$. (2)$S^2 \times \mathbb{E}^2$, $\mathbb{H}^2 \times S^2$ dominates no other geometry except those indicated in the domination diagram.*

Proof:

(1) follows from Table 3-1(4)(7), Corollary 6.1 and Proposition 6.2.

(2) Let M be of $S^2 \times \mathbb{E}^2$ or $\mathbb{H}^2 \times S^2$ geometry, thus is finitely covered by $\Sigma_g \times S^2$ ($g \geq 1$). Assume that N is geometric and the geometry of N is not $S^2 \times \mathbb{E}^2, \mathbb{H}^2 \times S^2$, nor $S^2 \times S^2, S^3 \times \mathbb{E}, \mathbb{CP}^2$. Note that all other geometries are contractible; N is a $K(G, 1)$. By Lemma 6.2, every map $\Sigma_g \times S^2 \to N$ is homotopic to a map $\Sigma_g \times S^2 \to \Sigma_g \to N$ where the first map is the projection. This map has zero degree for dimension reason. □

Proposition 6.8 *$\mathbb{E}^4, Nil \times \mathbb{E}, Sol \times \mathbb{E}$ and $\widetilde{SL_2\mathbb{R}} \times \mathbb{E}$ does not dominate $\mathbb{H}^2 \times S^2$.*

Proof: Let M be a typical covering manifold of any of these geometries. M has the form $N \times S^1$. $\pi_1(M)$ has non-trivial center, which must be mapped to zero by f_*. By Lemma 6.2, f_* can be homotopied to compress the S^1 factor of M, hence has zero degree. $\square$

Proposition 6.9 *$\mathbb{H}^3 \times \mathbb{E}$ does not dominate $Nil^4, Sol^4_{m,n}, Sol^4_0$ or Sol^4_1.*

Proof: Assume that $M_1 = N \times S^1$ is a typical covering manifold of $\mathbb{H}^3 \times \mathbb{E}$, and M_2 is any $Nil^4, Sol^4_{m,n}, Sol^4_0$ or Sol^4_1-manifold. $f : M_1 \to M_2$ is a map of nonzero degree. By taking a finite cover of M_2, we assume that f_* is surjective on fundamental groups.

For $Sol^4_{m,n}$ and Sol^4_0 case: By Proposition 4.3 and 4.5, $\pi_1(M_2)$ has trivial center. But $\pi_1(M_1)$ has nontrivial center, thus we have a contradiction.

For Nil^4 and Sol^4_1 case: By Proposition 4.12 and 4.11, M_2 is a circle bundle: $S^1 \to M_2 \to F$ where F is of Nil or Sol type. Thus $\pi_1(M_2)$ fits into an exact sequence: $0 \to \mathbb{Z} \to \pi_1(M_2) \to \pi_1(F) \to 0$. Furthermore, $\mathbb{Z}$ is the center of $\pi_1(M_2)$; and f_* must send $\pi_1(S^1) \in \pi_1(M)$ isomorphically onto $\mathbb{Z}$. Regarding M_1 as a trivial circle bundle over F, and by Lemma 6.2, f is homotopic to a bundle morphism f'. f' induces a map $g : N \to F$. g is of nonzero degree since f is (otherwise, g can be homotopied to a map from N to the 2-skeleton of F. By the homotopy lifting property, f' can be homotopied to the 3-skeleton of M_2).

Note that f' induces a bundle isomorphism between M_1 and g^*M_2, so we have $g^*e(M_2) = e(g^*M_2) = e(M_1) = 0$ where e denotes Euler class. By Proposition 4.12 and 4.11, $e(M_2)$ is non-torsion, hence is nonzero in $H^2(F, \mathbb{R})$. By Poincare duality, there exists $a \in H^1(F, \mathbb{R})$ such that $ae(M_2) = \lambda[F]$ $(\lambda \neq 0)$. However, $\lambda g^*([F]) = g^*(ae(M_2)) = g^*(a)g^*(e(M_2)) = 0$, so $\deg(g) = 0$. $\square$

7 Product Geometry Splitting

Inspired by Table 3-1 and its proof, we are able to prove the following generalized conclusion:

Proposition 7.1 *Let X be any geometry of dimension 2, 3 or 4, and n be any positive integer. Then every $X \times \mathbb{E}^n$-manifold is finitely covered by $N \times T^n$ where N is a X-manifold.*

Proof: Discuss case by case. First we fix our notation: G_0 is the structure group of $X \times \mathbb{E}^n$, G is the structure group of X. Γ is a discrete subgroup of G_0 acting freely on X such that $M = \Gamma \backslash X$ is compact. We have explained in section 3 that Γ is a lattice in G_0.

(1) $X = S^k (k = 2, 4)$:

$G_0 = SO(k+1) \times Iso^0(\mathbb{E}^n)$. Odd-dimensional orthogonal operator has fixed-point, hence the projection map $\Gamma \to Iso^0(\mathbb{E}^n)$ is an isomorphism. M is expressed as a S^k-bundle over a Euclidean n-manifold. Passing to a finite cover we can assume that the base manifold B is T^n, and now $\Gamma \cong \mathbb{Z}^n$. Let $g_i = (\phi_i, v_i)$ be a basis of Γ where $\phi_i \in SO(k+1)$.

As all the ϕ_i commutes, they are simultaneously diagonalized to the form

$$\begin{pmatrix} \cos a & -\sin a & \\ \sin a & \cos a & \\ & & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \cos a & -\sin a & & & \\ \sin a & \cos a & & & \\ & & \cos b & -\sin b & \\ & & \sin b & \cos b & \\ & & & & 1 \end{pmatrix}$$

This means that ϕ_i has common fixed point on S^k. This fixed point induces a section of M over B, and thus M is a trivial circle bundle over B. □

The case $X = S^2 \times S^2$ can be proved in the same way.

(2) $X = S^3$:

M is a smooth orbifold fibration over Euclidean n-orbifold with general fiber a S^3-manifold. Taking covering spaces first with respect to the base orbifold and then with respect to the fiber (the reader can check this), we can show that M is finitely covered by a S^3-bundle over T^n.

Next, the situation is analogous to (1). The difference is that ϕ_i does not necessarily has fixed point. But we can still diagonalize ϕ_i to the form:

$$\begin{pmatrix} \cos a_i & -\sin a_i & & \\ \sin a_i & \cos a_i & & \\ & & \cos b_i & -\sin b_i \\ & & \sin b_i & \cos b_i \end{pmatrix}$$

Let us fixed the basis of $\mathbb{R}^4$ such that ϕ_i has the form above. The following map induces a section of M over T^n:

$$\Sigma\lambda_i v_i \mapsto \frac{1}{\sqrt{2}}(\cos(\Sigma\lambda_i a_i), \sin(\Sigma\lambda_i a_i), \cos(\Sigma\lambda_i b_i), \sin(\Sigma\lambda_i b_i))$$

Hence M is a trivial S^3-bundle over T^n. □

The same method applies to the case $X = \mathbb{CP}^2$, where $G_0 = SU(3) \times Iso^0(\mathbb{E}^n)$.

(3) $X = \mathbb{H}^2, \mathbb{H}^3, \mathbb{H}^2 \times \mathbb{H}^2$:

In this case, $G_0 = G \times Iso(\mathbb{E}^n)$. G is semisimple. By Lemma 3.4 (for $n = 1, 2$) and Lemma 7.1 below (for $n > 2$; the group S is $G \times SO(n)$ here), $\Gamma' = \Gamma \cap Iso^0(\mathbb{E}^n)$ is a lattice (because $\Gamma \cap \mathbb{R}^n$ is a lattice), and the image H of the projection map $\Gamma \to G$ is also a lattice.

If Γ' contains only translation, then M is a Seifert T^n-bundle over H/X. Passing to a finite cover we can assume that H/X is smooth. The monodromy lies in $SO(n) \cap SL(\Gamma')$. This is a finite group (since $SL(\Gamma')$ is discrete, and any $x \in SO(n)$ has $||x||_2 = 1$). The rest of proof is the same as Table 3-1(5).

Suppose that Γ' contains a non-translation element g. The translation subgroup of Γ' (denote by Γ'') is of finite index. Denote by K the image of projection: $\Gamma \to Iso^0(\mathbb{E}^n)$. K fit into an exact sequence: $0 \to K' \to K \to K'' \to 0$ where $K' = K \cap \mathbb{R}^n$ and $K'' = K/K' \subset SO(n)$. From conjugation acions (K' on Γ', and K'' on Γ''), we can show that Γ'' is of finite index in K', and $K'' \subset SO(n) \cap SL(\Gamma'')$. Hence K is discrete. Then it is easy to show that $\Gamma \cap G$ is a lattice in G, and K/Γ' is finite. Finally, $\Gamma' \times (\Gamma \cap G)$ is a finite-index subgroup in Γ; M is finitely covered by ($\mathbb{E}^n$-manifold)$\times$(X-manifold). □

Lemma 7.1 ([9, Corollary 8.28]) *Let G be a connected Lie group, R be its radical. Let S be a semisimple subgroup of G such that $G = SR$. Denote by σ the conjugation action of S on R. Assume that $\ker(\sigma)$ has no compact factor in its identity component. Then if Γ is a lattice in G, $\Gamma \cap R$ is a lattice in R.* $\square$

(4) $X = \widetilde{SL_2\mathbb{R}}$:

$G_0 = (\widetilde{SL_2\mathbb{R}} \times_{\mathbb{Z}} \mathbb{R}) \times Iso^0(\mathbb{E}^n)$. Denote $N = \mathbb{R} \times Iso^0(\mathbb{E}^n)$, in which the $\mathbb{R}$ factor is the center of $Iso^0(\widetilde{SL_2\mathbb{R}})$. We claim that $\Gamma' = \Gamma \cap N$ is a lattice. The remaining part of the proof works in the same way as in (3) and Table 3-1(7).

The case $n = 1, 2$ follows from Lemma 3.4; it suffices to prove the case $n > 2$. The proof here is based on the proof of ([9, Theorem 8.27]). Denote $R = \mathbb{R} \times \mathbb{R}^n$, and by p_1 the projection map $G_0 \to G_0/R = Iso^0(\mathbb{H}^2) \times SO(n)$. Denote by p_2 the projection map $Iso^0(\mathbb{H}^2) \times SO(n) \to Iso^0(\mathbb{H}^2)$. Let U be the closure of $p_1(\Gamma)$. According to ([9, Theorem 8.24]), the identity component U^0 is solvable. The group $V = p_2(p_1(\Gamma)) \subset Iso^0(\mathbb{H}^2)$ has property (S) as defined in Chapter 5, [9]. For the same reason as in ([9, Theorem 8.27]), $p_2(U^0)$ is a normal subgroup of $Iso^0(\mathbb{H}^2)$, and is trivial. $\bar{V}^0$ has three possibilities: ① is trivial, then our claim is proven. ② $\bar{V}^0$ is a proper closed subgroup of $Iso^0(\mathbb{H}^2)$. Its Lie algebra $\mathfrak{v}$ is preserved by the adjoint action of V. By [9, Corollary 5.16(i)], $\mathfrak{v}$ is normal in $\mathfrak{iso}(\mathbb{H}^2)$, and is trivial. ③ $\bar{V}^0 = Iso^0(\mathbb{H}^2)$, i.e. V is dense. In addition that $SO(n)$ is compact, given any $p \in Iso^0(\mathbb{H}^2)$, there exists a sequence $g_i \in p_1(\Gamma)$ converging to (p, q) for some $q \in SO(n)$. This means that $p_2(U) = Iso^0(\mathbb{H}^2)$. On the other hand, since $U^0 \subset SO(n)$, $p_2(U) = p_2(p_1(\Gamma))$. Γ is finitely-generated, this is impossible. $\square$

(5) $X = \mathbb{H}^2 \times S^2$:

M is a S^2-bundle over a $\mathbb{H}^2 \times \mathbb{E}^n$-manifold, hence is covered by a S^2-bundle over $\Sigma_g \times T^n$. Let $a_i, b_i (i \leq i \leq g), e_i (1 \leq i \leq n)$ be the standard generetors of $\Sigma_g \times T^n$. Since $[a_i, e_j] = [b_i, e_j] = [e_i, e_j] = 1$, for the same reason as in (1), M has a section and is a trivial bundle. $\square$

(6) $X = Nil$:

The proof is quite similar to Table 3-1(8). In this case, $G_0 = (Nil \rtimes S^1) \times (\mathbb{R}^n \rtimes SO(n))$. From Lemma 7.1 (for $n > 2$) and Lemma 3.2 (for $n = 1, 2$) we know that $\Gamma \cap (Nil \times \mathbb{R}^n)$ is a lattice. The quotient group is finite; taking a finite cover we can assume that $\Gamma \subset Nil \times \mathbb{R}^n$. Working as in Table 3-1(8), we can then prove that $\Gamma \cap (\mathbb{R} \times \mathbb{R}^n)$ is a lattice. Hence M is a T^{n+1}-bundle over T^2. The monodromy is again trivial, and finally M splits as a product of T^n and a Nil-manifold. $\square$

The remaining cases are $Sol, Nil^4, Sol_0^4, Sol_{m,n}^4, Sol_1^4$. They are their own structure group, and are solvable. In these cases, $G_0 = G \times Iso^0(\mathbb{E}^n) = (G \times \mathbb{R}^n) \rtimes SO(n)$. Lemma 7.1 applies to all cases for $n > 2$: Denote $G' = G \times \mathbb{R}^n$, then $\Gamma \cap G'$ is a lattice. The quotient is a lattice in $SO(n)$, hence is finite. Replacing M by a finite cover, in the following paragraphs we assume that $\Gamma \subset G'$.

(When $n = 2$, use Lemma 3.2 instead and we can prove the same conclusion after some adjustment of the proofs below.)

(7) $X = Sol$:

$\Gamma \subset Sol \times \mathbb{R}^n$ is a lattice. Denote $N = V_1 \times V_2$ where $V_1 = \mathbb{R}^2$ is the nilradical of Sol, and $V_2 = \mathbb{R}^n$. By Lemma 3.2, $\Gamma \cap N$ is a lattice. Analogous to Table 3-1(9), we can show that there exists a finite-index sublattice $\Gamma' = \Gamma_1 \times \Gamma_2 \subset \Gamma$ such that Γ_i are lattices in V_i. Thus M is finitely covered by (Sol-manifold)$\times T^n$. $\square$

(8) $X = Sol_0^4, Sol_{m,n}^4$:
We have Proposition 4.3, 4.5. Then the proof is analogous to (7). $\square$

(9) $X = Nil^4$:
Now $\Gamma \subset Nil^4 \times \mathbb{R}^n$. Denote by $(e_1, e_2, e_3; t)$ the standard coordinates in Nil^4. The commutator subgroup is $N = \mathbb{R}(e_1, e_2)$. By Lemma 3.3, $\Gamma \cap N$ is a lattice. According to the conjugation relation (see section 4, part 1), we can assume that $l = (l_1, 0, 0; 0), h = (h_1, h_2, 0; 0)$ is a basis of $\Gamma \cap N$. $\Gamma' = \Gamma/(\Gamma \cap N) \subset \mathbb{R}^{2+n}$ is also a lattice. Assume that $g_i = (u_{i1}, u_{i2}, u_{i3}; t_i; v_{i1} \dots v_{in}) \in \Gamma$ are $n+2$ elements that projects to a basis of Γ'. Without losing generality, we can assume that $u_{13}t_2 - u_{23}t_1 \neq 0$. By some calculation,

$$[g_i, g_j] = (*, u_{j3}t_i - u_{i3}t_j, 0; 0; 0 \dots 0)$$

Thus $s_{ij} = u_{j3}t_i - u_{i3}t_j$ is an integral multiple of h_2. For $i = 3, 4 \dots n+2$, the element $g_i' = g_i^{s_{12}/h_2} g_2^{-s_2 i/h_2} g_1^{s_1 i/h_2}$ has the form $(w_{i1}, w_{i2}, 0; 0; * \dots *)$. Assume $t_1 \neq 0$; calculating $g_1 h g_1^{-1}$ and $g_1 g_i' g_1^{-1}$, we find that $h_2 t_1$ and $w_{i2} t_1$ are all integral multiples of l_1. Thus the ratio between h_2 and w_{i2} are all rational. Define $g_i'' = a g_i' + b h \, (a, b \in \mathbb{Z})$, they will have the form $g_i'' = (*, 0, 0; 0; * \dots *)$.

Denote by Γ_1 the lattice generated by $l, h, g_1, g_2, g_3'' \dots g_{n+2}''$. It is a finite-index sublattice of Γ. $g_3'' \dots g_{n+2}''$ freely generates an abelian subgroup $U_1 \subset \Gamma_1$. l, h, g_1, g_2 generates a subgroup U_2 which is isomorphic to a lattice in Nil^4. Clearly $\Gamma_1 = U_1 \times U_2$. Hence M is a product of Nil^4-manifold and T^n. $\square$

(10) $X = Sol_1^4$:
$\Gamma \subset Sol_1^4 \times \mathbb{R}^n$ is a lattice. In section 4 (part 4), we introduced the structure of Sol_1^4. By Lemma 3.2, $\Gamma' = \Gamma \cap (Nil \times \mathbb{R}^n)$ is a lattice. From our results in (6) and Table 3-1(8), we can assume that Γ' is generated by $z, x, y, v_1 \dots v_n$, where $z \in Z(Nil), [x, y] = z^k, v_i \in Z(Nil) \times \mathbb{R}^n$. $\Gamma/\Gamma' \subset Sol_1^4/Nil = \mathbb{Z}$ is also a lattice, and we suppose that $t \in \Gamma$ projects to the generator of it. Now $v_1 \dots v_n$ freely generates an abelian group V_1. z, x, y, t generates a group V_2 which is isomorphic to a lattice in Sol_0^4. Also, $\Gamma = V_1 \times V_2$. The product structure follows. $\square$

Finally, $X = \mathbb{E}^k$ follows from Bieberbach Theorem. $\square$

Corollary 7.2 *Suppose the dimension of X, Y are both 3 or 4. Then $X \to Y$ implies $X \times \mathbb{E}^n \to Y \times \mathbb{E}^n$.* $\square$

It is natural to may ask whether the converse holds. The answer is affirmative.

Proposition 7.3 *Suppose M_1, M_2 are two manifolds of the same dimension. M_2 is a $K(G, 1)$ and $\pi_1(M_2)$ is torsion-free. If $M_1 \times S^1$ dominates $M_2 \times S^1$, then some finite cover of M_1 dominates M_2.*

Proof: Let $f : M_1 \times S^1 \to M_2 \times S^1$ be a nonzero degree map. Denote by t_i the generator of $\mathbb{Z} = \pi_1(S^1) \subset \pi_1(M_i \times S^1)$. $f_*(t_1)$ is nonzero: otherwise from Lemma 6.2, f can be homotopied to factor through $M_1 \times \{pt\}$, and cannot have nonzero degree. Since $\pi_1(M_2)$ is torsion-free, we have $\ker(f_*) \subset \pi_1(M_1)$. Let $H \subset \pi_1(M_2 \times S^1)$ be the image of f_*; clearly H has finite index. Assume that the projection map $H \to \mathbb{Z}t_2$ is surjective (otherwise we replace the S^1 factor in $M_2 \times S^1$ by a finite cover of it).

Let m be the smallest integer such that $t_2' = mt_2$ lies in H. Denote $I = H \cap \pi_1(M_2)$. Then $I \times \mathbb{Z}t_2'$ has finite index in H (Lemma 3.5). On the other hand: define $L = \pi_1(M_1)/\ker(f_*)$, then $H = L \times \mathbb{Z}(f_*(t_1))$.

If $f_*(t_1) \neq t_2'$, then we use the following method to reduce to the case $f_*(t_1) = t_2'$. First we claim that $H' = (I \cap L) \times (\mathbb{Z}(f_*(t_1)) \times \mathbb{Z}t_2')$ is of finite index in H. This is clearly a direct product. We have an exact sequence $1 \to IL/(I \cap L) \to H/(I \cap L) \to H/IL \to 1$. IL contains L and is not equal to L, thus is of finite index in H. There exists $a, b \in \mathbb{Z}$ such that $af_*(t_1) \in IL, bt_2' \in IL$. $IL/(I \cap L)$ is isomorphic to $(IL/I) \times (IL/L)$, and $af_*(t_1), bt_2'$ generates a finite-index subgroup of it. Our claim then follows.

Then we take a finite cover of $M_1 \times S^1$ with respect to $f_*^{-1}(H')$. Since $f_*(t_1) \in H'$, the covering map has the form $p \times id : M_1' \times S^1 \to M_1 \times S^1$. $K(H', 1)$ can be regarded as a covering space of $M_2 \times S^1$; it is a manifold. There is an automorphism of H' exchanging $f_*(t_1)$ and t_2'. This map induces a self homotopy equivalence $\phi : K(H', 1) \to K(H', 1)$ which has degree -1. Define a composed map $F : M_1' \times S^1 \to K(H', 1) \to K(H', 1) \to M_2 \times S^1$, where the first map is induced by f, the second map is the ϕ defined above, and the third map is the natural projection. From our construction, $\deg(F)$ is nonzero, and F_* maps t_1 to t_2'. Replacing M_1 by M_1', and f by F, we reduce to the case $f_*(t_1) = t_2'$.

Now the homomorphism f_* can be represented by a matrix:

$$\begin{pmatrix} f_{*1} : \pi_1(M_1) \to \pi_1(M_2) & 0 : \pi_1(S^1) \to \pi_1(M_2) \\ f_{*2} : \pi_1(M_1) \to \pi_1(S^1) & (\times m) : \pi_1(S^1) \to \pi_1(S^1) \end{pmatrix}$$

By Lemma 6.2, f is homotopic to the following matrix of maps:

$$\begin{pmatrix} f_1 : M_1 \to M_2 & pt : S^1 \to M_2 \\ f_2 : M_1 \to S^1 & g : S^1 \to S^1 \end{pmatrix}$$

From the matrix, we have $\deg(f) = \pm \deg(f_1) \deg(g)$. $\deg(f)$ and $\deg(g)$ are both nonzero, hence $\deg(f_1)$ is. Our proposition follows. $\square$

Proposition 7.4 *Suppose that X, Y are geometries, and has dimension 3 or 4. in addition, $X \neq \mathbb{H}^2 \times \mathbb{H}^2$ when Y is not contractible. Then $X \times \mathbb{E}^n \to Y \times \mathbb{E}^n$ implies $X \to Y$.*

remark: The additional requirement is because we do not know whether $\mathbb{H}^2 \times \mathbb{H}^2$ dominates non-contractable geometries (such as $\mathbb{H}^2 \times S^2$). Note that Proposition 7.3 is not available in this case.

Proof: By Proposition 7.1, all $X \times \mathbb{E}^n$-manifolds are covered by $N \times T^n$, where N is a typical covering X-manifold. Checking case by case (see Table 3-1), we can confirm that the fundamental group of all typical covering X-manifolds are torsion-free. If Y is contractible, then the conclusion follows from Proposition 7.3 and induction by n.

The remaining cases in which Y is not contractible can be implied by one of the terms below:

①: $S^2 \times S^2 \times \mathbb{E}^n$ and $S^3 \times \mathbb{E}^{n+1}$ does not dominate each other. It suffices to prove $S^2 \times S^2 \times T^n$ and $S^3 \times T^{n+1}$ does not dominate each other. This easily follows from cohomology ring computation.

②: $\mathbb{CP}^2 \times \mathbb{E}^n \nrightarrow S^2 \times S^2 \times \mathbb{E}^n$. It suffices to prove $\mathbb{CP}^2 \times T^n$ does not dominate $S^2 \times S^2 \times T^n$. It is also easy from cohomology.

③: $Nil^4 \times \mathbb{E}^n$ does not dominate $S^2 \times \mathbb{E}^{n+2}$. Cohomology computation can be found in Proposition 6.5(2).

④: $(Sol^4_{m,n}, Sol^4_0, Sol^4_1) \times \mathbb{E}^n$ does not dominate $\mathbb{CP}^2 \times \mathbb{E}^n$. Cohomology computation has been done in Proposition 4.4, 4.7 and 4.10.

⑤: $\mathbb{E}^{n+4}, Sol \times \mathbb{E}^{n+1}$ and $\widetilde{SL_2\mathbb{R}} \times \mathbb{E}^{n+1}$ do not dominate $\mathbb{H}^2 \times S^2 \times \mathbb{E}^n$. The typical covering manifolds of these geometries all have the form $M = N \times T^r (r > n)$. Every map from M to $\Sigma_g \times S^2 \times T^n$ can be homotopied to compress at least one S^1 factor, hence has zero degree for dimension reason. □

The following main theorem will be proved once we complete the discussion of $\mathbb{H}^2 \times \mathbb{H}^2$.

Corollary 7.5 *Let X, Y be geometries of (the same) dimension 2,3,4. Then $X \to Y$ if and only if $X \times \mathbb{E}^n \to Y \times \mathbb{E}^n$.*

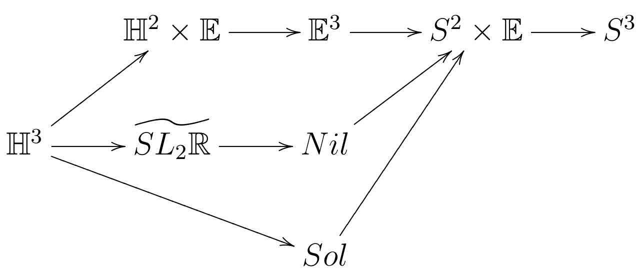
8 Proof of Diagram 1-1: Non-arrows (Part 2)

We have come to the final step of the proof of Diagram 1-1. What is left is the non-arrows in Diagram 1-2, which implies the remaining non-arrows of Diagram 1-1.

Lemma 8.1 ([10, Proposition 2.1]) *Assume that M and N are two 3-dimensional Seifert bundles which are both $K(G,1)$. $f : M \to N$ is of nonzero degree. Then $e(M)$ and $e(N)$ are either both nonzero or both zero.* □

Proof of Proposition 2.3:

Recall Diagram 1-2:



We need only prove:

(1) $S^3 \nrightarrow S^2 \times \mathbb{E}$. This is clear.

(2) $S^2 \times \mathbb{E}$ does not dominate any geometry except for S^3. The method in Proposition 6.7 applies here.

(3) $Sol \nrightarrow \mathbb{E}^3$ because the first betti number of Sol-manifold is smaller than 3. $Sol \nrightarrow Nil$ for the same reason as in the proof of Proposition 6.5(3).

(4) There are no more arrows between $\mathbb{H}^2 \times \mathbb{E}, \mathbb{E}^3, \widetilde{SL_2\mathbb{R}}$ and Nil. The reason is Lemma 8.1. The conclusion is even stronger: any two manifolds having geometries that are not connected by arrows does not have domination relation.

(5) $\mathbb{H}^2 \times \mathbb{E}, \widetilde{SL_2\mathbb{R}}$ does not dominate Sol. This is because the fundamental group of Sol-manifold has trival center. □

Corollary 8.1 *Let X, Y be any 3-dimensional geometries. Except for those indicated in Diagram 1-1, there is no more arrows between $X \times \mathbb{E}$ and $Y \times \mathbb{E}$.* □

With Corollary 8.1 and the propositions in section 6, we have completed the proof of non-arrows in Diagram 1-1.

9 Results on $\mathbb{H}^2 \times \mathbb{H}^2$

In this section, we treat the last case: $\mathbb{H}^2 \times \mathbb{H}^2$. Our previous results are mainly based on virtual splitting. However, not all $\mathbb{H}^2 \times \mathbb{H}^2$-manifold virtually split into a product of surfaces.

An $\mathbb{H}^2 \times \mathbb{H}^2$-manifold M is called **reducible** if it is finitely covered by $\Sigma_{g_1} \times \Sigma_{g_2}$. Otherwise it is called **irreducible**. The fundamental group of a reducible manifold is commensurable with a product of Fuchsian groups. The fundamental group of an irreducible manifold is arithmetic, in the sense of [4, §9.5].

We separate $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds into two class: reducible ones and irreducible ones. Manifolds belonging to different classes are not the covering of each other. The dominating ability of these two classes of manifolds are quite different.

Reducible $\mathbb{H}^2 \times \mathbb{H}^2$ clearly dominates $\mathbb{H}^2 \times \mathbb{E}^2$. For non-domination, we have the following result:

Proposition 9.1 *$\Sigma_{g_1} \times \Sigma_{g_2}$ does not dominate any $Sol \times \mathbb{E}, Nil^4, Sol^4_{m,n}, Sol^4_0, Sol^4_1$-manifold, as well as any typical covering $Nil \times \mathbb{E}$-manifold.*

Proof:

Suppose that $f : M_1 \to M_2$ has nozero degree, where $M_1 = \Sigma_{g_1} \times \Sigma_{g_2}$, M_2 is any $Sol \times \mathbb{E}, Nil^4, Sol^4_{m,n}, Sol^4_0$ or Sol^4_1-manifold, or any typical covering manifold of $Nil \times \mathbb{E}$. By Lemma 6.1, we can take a finite cover of M_2 such that f_* is surjective on fundamental groups. Proposition 3.1 guarantees that this causes no problem in the case of $Nil \times \mathbb{E}$. By Proposition 3.1, 3.2, 4.1, 4.3, 4.5 and 4.8, $\pi_1(M_2)$ has the form $1 \to N \to \pi_1(M_2) \to \mathbb{Z} \to 0$ where N is abelian or nilpotent.

Writting $\pi_1(M_2)$ in the form $\pi_1(M_2) = N \rtimes_A \mathbb{Z}$, the action A is described in the conclusions or proofs of the Propositions. We do not use the concrete form of A here, only note that A is non-trivial.

Consider the subgroup $f_*^{-1}(N) \subset \pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}) = \pi_1(M_1)$, and denote by H_i the image of the projection map $f_*^{-1}(N) \subset \pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}) \to \pi_1(\Sigma_{g_i})$. There are two cases:

(1): One of the H_i, say H_1, is of infinite index in $\pi_1(\Sigma_{g_1})$. But there is an exact sequence $1 \to f_*^{-1}(N) \to \pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}) \to \mathbb{Z} \to 0$ induced from the exact sequence of $\pi_1(M_2)$. Now $f_*^{-1}(N) \subset H_1 \times \pi_1(\Sigma_{g_2})$, and the only possibility is that $f_*^{-1}(N) = H_1 \times \pi_1(\Sigma_{g_2})$ and $\pi_1(\Sigma_{g_2})/H_1 = \mathbb{Z}$. Therefore, $\pi_1(\Sigma_{g_2})$ is mapped by f_* into N.

In the cases of $Nil \times \mathbb{E}, Sol \times \mathbb{E}, Nil^4, Sol^4_{m,n}$ and Sol^4_0, $N = \mathbb{Z}^3$. By Lemma 6.2, f is homotopic to a composed map $f' : \Sigma_{g_1} \times \Sigma_{g_2} \to \Sigma_{g_1} \times T^3 \to M_2$; we denote the latter map $\Sigma_{g_1} \times T^3 \to M_2$ by g. In the case of $Nil \times \mathbb{E}$, the center of $\pi_1(M_2)$ is isomorphic to $\mathbb{Z}^2$.

Hence by a further homotopy, g compresses one S^1 factor of T^3 and thus factors through $\Sigma_{g_1} \times T^2$. Proposition 7.3 and 2.3 now tells us that $\deg(g)$ must be zero. In other cases, the center of $\pi_1(M_2)$ is either trivial or is isomorphic to $\mathbb{Z}$, g_* must send $\pi_1(T^3) = \mathbb{Z}^3$ to zero or $\mathbb{Z}$. Anyway, g is homotopic to a map $\Sigma_{g_1} \times T^3 \to \Sigma_{g_1} \times S^1 \to M_2$, which has zero degree for dimension reason. In all situations, we have $\deg(f) = 0$.

In the case of Sol_1^4, N is the fundamental group of a Nil-manifold E and can be presented as $N =< z, x, y : [z, x] = [z, y] = 1, [x, y] = z^k >$. z generates the center of N, which is also the center of $\pi_1(M_2)$. f_* factors through N, hence by Lemma 6.2, f is homotopic to a composed map $f' : \Sigma_{g_1} \times \Sigma_{g_2} \to \Sigma_{g_1} \times E \to M_2$; we denote the latter map by g. To avoid confusion, we let $\pi_1(E)$ be presented as $\pi_1(E) =< z', x', y' : [z', x'] = [z', y'] = 1, [x', y'] = z'^k >$, which is isomorphic to N. By construction, g_* maps z' to z. Obviously $\pi_1(\Sigma_{g_1} \times E)/\mathbb{Z}z' \cong \pi_1(\Sigma_{g_1}) \times \mathbb{Z}^2$. By Proposition 4.11, $\pi_1(M_2)/\mathbb{Z}z := L$ is isomorphic to the fundamental group of a Sol-manifold. Taking this quotient on both sides, g_* induces a homomorphism $\bar{g}_* : \pi_1(\Sigma_{g_1}) \times \mathbb{Z}^2 \to L$. However, being the fundamental group of a Sol-manifold, L has trivial center. Hence $\mathbb{Z}^2 \subset \ker(\bar{g}_*)$. Returning to g_*, this means that g_* maps x' and y' into $\mathbb{Z}z$. Then $z^k = g_*(z'^k) = g_*([x', y']) = [g_*(x'), g_*(y')] = 1$ which is a contradiction.

(2): Both H_i has finite index in $\pi_1(\Sigma_{g_i})$. Take a finite cover $M_1' \to M_1$ corresponding to $H_1 \times H_2$, and still denote by f the composed map $M_1' \to M_2$. $f_*(\pi_1(M_1'))$ still contains $N \in \pi_2(M_2)$. We take a further cover $M_2' \to M_2$ corresponding to the image $f_*(\pi_1(M_1'))$. Now $\pi_1(M_2')$ fits into the exact sequence $1 \to N \to \pi_1(M_2') \to \mathbb{Z}m \to 0$, and we have an induced map $f' : M_1' \to M_2'$ which has the same degree as f. f_*' is still surjective on fundamental groups. M_1' still has the form of $\Sigma_{g_1'} \times \Sigma_{g_2'}$. $f_*^{-1}(N)$ is not changed. We now replace M_1 by M_1' and M_2 by M_2', and inherit all the notations defined above.

The difference is that now we have $H_i = \pi_1(\Sigma_{g_i})$. Define $I_i = f_*^{-1}(N) \cap \pi_1(\Sigma_{g_i})$. Clearly I_i are normal subgroups of $\pi_1(\Sigma_{g_i})$. According to Lemma 3.5, we have an isomorphism $f_*^{-1}(N)/(I_1 \times I_2) \cong \pi_1(\Sigma_{g_1})/I_1 \cong \pi_1(\Sigma_{g_2})/I_2 \cong (\pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}))/f_*^{-1}(N) \cong \mathbb{Z}$. If $x = (x_1, x_2) \in f_*^{-1}(N)$ projects to a generator of $f_*^{-1}(N)/(I_1 \times I_2)$, then $x' = (x_1, 1)$ projects to a generator of $(\pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}))/f_*^{-1}(N)$. Clearly $f_*(x') \notin N$ and its image generates $\pi_1(M_2)/N = \mathbb{Z}$. Also, $x_i (i = 1, 2)$ projects to the generator of $\pi_1(\Sigma_{g_i})/I_i \cong \mathbb{Z}$ (see the proof of Lemma 3.5).

We claim that $f_*(x) \neq 1$. Otherwise, for any $y \in \pi_1(\Sigma_{g_1})$, $f_*((y, 1)x(y^{-1}, 1)) = f_*(yx_1y^{-1}, x_2) = 1$. Miltiplying by x^{-1} we get $[x_1, \pi_1(\Sigma_{g_1})] \subset \ker(f_*)$. Analogously, $[x_2, \pi_1(\Sigma_{g_2})] \subset \ker(f_*)$. $f_*(x) = 0$ also implies that f_* sends $I_1 \times I_2$ surjectively onto N. For any $i = (i_1, i_2) \in I_1 \times I_2$, $x'(i_1, i_2)x'^{-1} = (x_1 i_1 x_1^{-1}, i_2) = (i_1, i_2)j$ where $j \in \ker(f_*)$, but $f_*(x')f_*(i)f_*(x'^{-1})$ is not always equal to $f_*(i)$, because $\pi_1(M_2) = N \rtimes_A \mathbb{Z}$ and A is not trivial. This leads to a contradiction.

Now since $f_*(x) \neq 1$, $\ker(f_*) \subset I_1 \times I_2$. We wish to determine the quotient group $(I_1 \times I_2)/\ker(f_*)$. This is a normal subgroup of $f_*^{-1}(N)/\ker(f_*) \cong N$ with quotient $f_*^{-1}(N)/(I_1 \times I_2) \cong \mathbb{Z}$. Depending on the geometry of M_2, N is isomorphic to $\mathbb{Z}^3$ or $\mathbb{Z}^2 \rtimes_C \mathbb{Z}$(C is idempotent). Anyway, $(I_1 \times I_2)/\ker(f_*)$ must be isomorphic to $\mathbb{Z}^2$.

Denote by J_i the intersection $\ker(f_*) \cap I_i$, and by P_i the image of the projection: $\ker(f_*) \hookrightarrow I_1 \times I_2 \to I_i$. For any $a \in P_1$ and $b \in \pi_1(\Sigma_{g_1})$, by definition, there exists $c \in I_2$ such that $f_*((a, c)) = 1$. Also $1 = f_*((b, 1)(a, c)(b, 1)^{-1}) = f_*((bab^{-1}, c))$. Multiplying by $(a^{-1}, c^{-1}) \in \ker(f_*)$, we have the conclusion $[P_1, \pi_1(\Sigma_{g_1})] \subset \ker(f_*)$. The conclusion for P_2 is analogous.

By Lemma 3.5, $\ker(f_*)/(J_1 \times J_2) \cong P_1/J_1 \cong P_2/J_2 \cong (P_1 \times P_2)/\ker(f_*)$. The last group is a subgroup of $(I_1 \times I_2)/\ker(f_*) \cong \mathbb{Z}^2$. Hence there are three cases:

(a) $(P_1 \times P_2)/\ker(f_*) = 0$. This means that $\ker(f_*) = P_1 \times P_2$ and $(I_1/P_1) \times (I_2/P_2) \cong \mathbb{Z}^2$. Hence either $I_1/P_1 \cong I_2/P_2 \cong \mathbb{Z}$, or $I_1 = P_1$, $I_2/P_2 \cong \mathbb{Z}^2$.

For the former case, $\pi_1(\Sigma_{g_1})/P_1$ must be isomorphic to $\mathbb{Z}^2$ or the Klein bottle group since there is an exact sequence

$$0 \to \mathbb{Z}(= I_1/P_1) \to \pi_1(\Sigma_{g_1})/P_1 \to \mathbb{Z}(= \pi_1(\Sigma_{g_1})/I_1) \to 0$$

So does $\pi_2(\Sigma_{g_1})/P_2$. If both are $\mathbb{Z}^2$, then f_* factors through $\mathbb{Z}^2 \times \mathbb{Z}^2$ which is abelian, and cannot have finite-index image. If any one $\pi_1(\Sigma_{g_i})/P_i$ is the Klein bottle group, then we take a double cover of Σ_{g_i} and can derive the same contradiction.

For the latter case, $\pi_1(\Sigma_{g_1})/P_1 = \pi_1(\Sigma_{g_1})/I_1 \cong \mathbb{Z}$. By Lemma 6.2, f is homotopic to a map $f' : \Sigma_{g_1} \times \Sigma_{g_2} \to S^1 \times \Sigma_{g_2} \to M_2$ which has zero degree for dimension reason.

(b) $(P_1 \times P_2)/\ker(f_*) \cong \mathbb{Z}$. This means that $(I_1/P_1) \times (I_2/P_2) = (I_1 \times I_2)/(P_1 \times P_2) \cong \mathbb{Z}$, which implies that one of the groups I_i/P_i, say I_1/P_1, is trivial. As explained above, $\mathbb{Z} \cong (P_1 \times P_2)/\ker(f_*) \cong P_1/J_1$, so $I_1/J_1 \cong \mathbb{Z}$. Again we have $\pi_1(\Sigma_{g_1})/J_1 \cong \mathbb{Z}^2$ (or the Klein bottle group; and we take double cover). By Lemma 6.2, f is homotopic to a map $f' : \Sigma_{g_1} \times \Sigma_{g_2} \to T^2 \times \Sigma_{g_2} \to M_2$. Denote by g the latter map. For $Sol \times \mathbb{E}, Nil^4, Sol^4_{m,n}, Sol^4_0, Sol^4_1$ geometry, the center of $\pi_1(M_2)$ is either trivial or isomorphic to $\mathbb{Z}$. Hence after a homotopy, g compresses one S^1 factor of T^2, therefore has zero degree for dimension reason. For $Nil \times \mathbb{E}$ geometry, Proposition 7.3 and Proposition 2.3 tells us that $\deg(g) = 0$.

(c) $(P_1 \times P_2)/\ker(f_*) \cong \mathbb{Z}^2$. This means that $P_1 = I_1$ and $P_2 = I_2$. From the conclusion above, $[I_i, \pi_1(\Sigma_{g_i})] \subset \ker(f_*)$ $(i = 1, 2)$. As a smaller subgroup, $[I_i, \mathbb{Z}x_i] \subset \ker(f_*)$. Now we need another simple lemma:

Lemma 9.1 *Let G be a group, H_1, H_2, H_3 be subgroups. Assume that H_2 is normal. Then we have $[H_1, H_2H_3] \subset [H_1, H_2]N([H_1, H_3])$. The notation $N(H)$ denotes the smallest normal subgroup of G containing H (abuse of notation).*

Proof: It is easy to check that for any $h_i \in H_i$, $[h_1, h_2h_3] = [h_1, h_2]h_2[h_1, h_3]h_2^{-1}$. □

Proposition 9.1 cont'd: It is easy to see that $[\pi_1(\Sigma_{g_i}), \mathbb{Z}x_i] = [I_i\mathbb{Z}x_i, \mathbb{Z}x_i] = [I_i, \mathbb{Z}x_i] \subset \ker(f_*)$. Recall that I_i is normal in $\pi_1(\Sigma_{g_i})$. Use Lemma 9.1, we have $[\pi_1(\Sigma_{g_i}), \pi_1(\Sigma_{g_i})] = [\pi_1(\Sigma_{g_i}), I_i\mathbb{Z}x_i] \subset \ker(f_*)$ ($\ker(f_*)$ is normal). Thus f_* factors through a map $\pi_1(\Sigma_{g_1}) \times \pi_1(\Sigma_{g_2}) \to \mathbb{Z}^{2g_1+2g_2} \to \pi_1(M_2)$ which cannot be surjective. We get a contradiction. □

Corollary 9.2 *$\Sigma_{g_1} \times \Sigma_{g_2}$ does not dominate any $Nil \times \mathbb{E}$-manifold.*

Proof: Any finite cover of $\Sigma_{g_1} \times \Sigma_{g_2}$ is finitely covered by $\Sigma_{g'_1} \times \Sigma_{g'_2}$ for some g'_1, g'_2. □

The dominating ability of irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ is much weaker:

Proposition 9.3 *Let M be a irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold. Then M dominates $S^2 \times S^2$.*

Proof: According to Matsushima's result in [8], M satisfies $b^2(M) = 2 + b^{(2,0)}(M)$, where $b^{(2,0)}$ is the dimension of $H^{(2,0)}(M)$ (with respect to the natural Kahler structure on M). Simple calculation of the dimension of cohomology groups leads to the conclusion $\sigma(M) = 0$ (Wall has an elegant explanation of this in [17, §6]). It is well-known that any indefinite unimodular quadratic form on an integral lattice represents zero (see [11, §V.2]). Hence M satisfies the conditions of Propositon 5.1. □

Proposition 9.4 *Let M be a irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold. Then M does not dominate any geometric manifold except $\mathbb{CP}^2$, $S^2 \times S^2$ and its quotients.*

Proof: The Margulis Normal subgroup Theorem (see [7, Chapter IV]) implies that any normal subgroup of $\pi_1(M)$ has finite index.

Let M' be a typical covering manifold of any geometry except $\mathbb{CP}^2$ and $S^2 \times S^2$; we have $b_1(M') > 0$ (check the geometries one by one). By Margulis's Theorem, M cannot dominate M', otherwise the inverse image of $[\pi_1(M'), \pi_1(M')]$ will be an infinite-index normal subgroup of $\pi_1(M)$. □

In fact, all domination relations between irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds are trivial, i.e. M dominates N iff M covers N. This follows from Margulis's Theorem and Mostow Rigidity.

One can easily see that no geometry dominates $\mathbb{H}^2 \times \mathbb{H}^2$. The other geometries are wither solvable, or has S^2 or S^3 factors, or has non-trivial center. The $\mathbb{H}^2 \times \mathbb{H}^2$-part of Diagram 1-1 is thus complete.

Completion of proof of Theorem 7.5: Except for what we have proved in Proposition 7.4, we only need to prove: When $X = \mathbb{H}^2 \times \mathbb{H}^2$ and Y is non-contractible, $X \not\twoheadrightarrow Y$ implies $X \times \mathbb{E}^n \not\twoheadrightarrow Y \times \mathbb{E}^n$.

It suffices to prove $M \times T^n$ does not dominate $S^3 \times T^{n+1}$, where M is any irreducible $\mathbb{H}^2 \times \mathbb{H}^2$-manifold. For the same reason as in Proposition 9.4, we can prove that $M \times T^n$ dominates another manifold M_1 only if $b_1(M_1) \leq n$. □

Remark: We can regard the reducible and irreducible $\mathbb{H}^2 \times \mathbb{H}^2$ as two different geometries. In this case, the reducible one dominates $\mathbb{H}^2 \times \mathbb{E}^2$ and the irreducible one dominates $S^2 \times S^2$. If we regard $\mathbb{H}^2 \times \mathbb{H}^2$ as one geometry, it can only dominate $S^2 \times S^2$. Proposition 7.5 holds for both cases.

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